Operation Research

MODULE-3

3.1 <u>Introduction to Transportation Problem</u> The Transportation problem is to transport various amounts of a single homogeneous commodity that are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum.

It can also be defined as to ship goods from various origins to various destinations in such a manner that the transportation cost is a minimum.

The availability as well as the requirements is finite. It is assumed that the cost of shipping is linear.

Mathematical Formulation

Let there be m origins, i^{th} origin possessing a_i units of a certain product

Let there be n destinations, with destination j requiring b_j units of a certain product

Let c_{ij} be the cost of shipping one unit from i^{th} source to j^{th} destination

Let x_{ij} be the amount to be shipped from i^{th} source to j^{th} destination

It is assumed that the total availabilities Σa_i satisfy the total requirements Σb_j i.e.

$\Sigma a_i = \Sigma b_j$ (i = 1, 2, 3 … m and j = 1, 2, 3 …n)

The problem now, is to determine non-negative of x_{ij} satisfying both the availability constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad \text{for } i = 1, 2, .., m$$

as well as requirement constraints

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \text{for } j = 1, 2, .., n$$

and the minimizing cost of transportation (shipping)

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\, c_{ij} \quad \text{(objective function)}$$

This special type of LPP is called as **Transportation Problem**.

Tabular Representation

Let 'm' denote number of factories (F_1, F_2 … F_m)

Let 'n' denote number of warehouse (W_1, W_2 …

W_n) W→

F ↓	W_1	W_2	…	W_n	Capacities (Availability)
F_1	c_{11}	c_{12}	…	c_{1n}	a_1
F_2	c_{21}	c_{22}	…	c_{2n}	a_2
.	.	.	.	.	.
.	.	.	.	.	.
F_m	c_{m1}	c_{m2}	…	c_{mn}	a_m
Required	b_1	b_2	…	b_n	$\Sigma a_i = \Sigma b_j$

W→ F ↓	W_1	W_2	…	W_n	Capacities (Availability)
F_1	x_{11}	x_{12}	…	x_{1n}	a_1
F_2	x_{21}	x_{22}	…	x_{2n}	a_2
.	.	.	.	.	.
.	.	.	.	.	.
F_m	x_{m1}	x_{m2}	…	x_{mn}	a_m
Required	b_1	b_2	…	b_n	$\Sigma a_i = \Sigma b_j$

In general these two tables are combined by inserting each unit cost c_{ij} with the corresponding amount x_{ij} in the cell (i, j). The product $c_{ij}\, x_{ij}$ gives the net cost of shipping units from the factory F_i to warehouse W_j.

Some BasiDefinitions

- **Feasible Solution**
  A set of non-negative individual allocations ($x_{ij} \geq 0$) which simultaneously removes deficiencies is called as feasible solution.

- **Basic Feasible Solution**

A feasible solution to 'm' origin, 'n' destination problem is said to be basic if the number
of positive allocations are m+n-1. If the number of allocations is less than m+n-1 then it is called as **Degenerate Basic Feasible Solution**. Otherwise it is called as Non- Degenerate Basic Feasible Solution.

- **Optimum Solution**
  A feasible solution is said to be optimal if it minimizes the total transportation cost.

Methods for Initial Basic Feasible Solution

Some simple methods to obtain the initial basic feasible solution are

1. North-West Corner Rule
2. Row Minima Method
3. Column Minima Method
4. Lowest Cost Entry Method (Matrix Minima Method)
5. Vogel's Approximation Method (Unit Cost Penalty Method)

North-West Corner Rule

Step 1

- The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the table.
- The maximum possible amount is allocated here i.e. $x_{11} = \min(a_1, b_1)$. This value of x_{11} is then entered in the cell (1,1) of the transportation table.

Step 2

i. If $b_1 > a_1$, move vertically downwards to the second row and make the second allocation of amount $x_{21} = \min(a_2, b_1 - x_{11})$ in the cell (2, 1).

ii. If $b_1 < a_1$, move horizontally right side to the second column and make the second allocation of amount $x_{12} = \min(a_1 - x_{11}, b_2)$ in the cell (1, 2).

iii. If $b_1 = a_1$, there is tie for the second allocation. One can make a second allocation of magnitude $x_{12} = \min(a_1 - a_1, b_2)$ in the cell (1, 2) or $x_{21} = \min(a_2, b_1 - b_1)$ in the cell (2, 1)

Find the initial basic feasible solution by using North-West Corner Rule

1.

W→ F↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Solution

	W_1	W_2	W_3	W_5	Availability
F_1	5 (19)	2 (30)			7 2 0
F_2		6 (30)	3 (40)		9 3 0
F_3			4 (70)	14 (20)	18 14 0
Requirement	5 0	8 6 0	7 4 0	14 0	

Initial Basic Feasible Solution

$x_{11} = 5$, $x_{12} = 2$, $x_{22} = 6$, $x_{23} = 3$, $x_{33} = 4$, $x_{34} = 14$

The transportation cost is 5 (19) + 2 (30) + 6 (30) + 3 (40) + 4 (70) + 14 (20) = Rs. 1015

2.

	D_1	D_2	D_3	D_4	Supply
O_1	1	5	3	3	34
O_2	3	3	1	2	15
O_3	0	2	2	3	12
O_4	2	7	2	4	19
Demand	21	25	17	17	80

Solution

	D_1	D_2	D_3	D_4	Supply
O_1	21 (1)	13 (5)			34 13 0
O_2		12 (3)	3 (1)		15 3 0
O_3			12 (2)		12 0
O_4			2 (2)	17 (4)	19 17
Demand	21 0	25 12 0	17 14 2 0	17 0	

Initial Basic Feasible Solution
$x_{11} = 21$, $x_{12} = 13$, $x_{22} = 12$, $x_{23} = 3$, $x_{33} = 12$, $x_{43} = 2$, $x_{44} = 17$
The transportation cost is 21 (1) + 13 (5) + 12 (3) + 3 (1) + 12 (2) + 2 (2) + 17 (4) = Rs. 221

3.

From	To					Supply
	2	11	10 4	3	7	
	1	4	7	2	1	8
	3	1	4	8	12	9
Demand	3	3	4	5	6	

Solution

From	To					Supply
	3 (2)	1 (11)				4 1 0
		2 (4)	(7)	(2)		8 6 2 0
				3 (8)	(12)	9 6 0
Demand	3 0	3 2 0	4 0	5 3 0	6 0	

Initial Basic Feasible Solution
$x_{11} = 3$, $x_{12} = 1$, $x_{22} = 2$, $x_{23} = 4$, $x_{24} = 2$, $x_{34} = 3$, $x_{35} = 6$
The transportation cost is 3 (2) + 1 (11) + 2 (4) + 4 (7) + 2 (2) + 3 (8) + 6 (12) = Rs. 153

Row Minima Method

Step 1

- The smallest cost in the first row of the transportation table is determined.
- Allocate as much as possible amount $x_{ij} = \min(a_1, b_j)$ in the cell (1, j) so that the capacity of the origin or the destination is satisfied.

Step 2

- If $x_{1j} = a_1$, so that the availability at origin O_1 is completely exhausted, cross out the first row of the table and move to second row.
- If $x_{1j} = b_j$, so that the requirement at destination D_j is satisfied, cross out the j^{th} column and reconsider the first row with the remaining availability of origin O_1.
- If $x_{1j} = a_1 = b_j$, the origin capacity a_1 is completely exhausted as well as the requirement at destination D_j is satisfied. An arbitrary tie-breaking choice is made. Cross out the j^{th} column and make the second allocation $x_{1k} = 0$ in the cell (1, k) with c_{1k} being the new minimum cost in the first row. Cross out the first row and move to second row.

Step 3
Repeat steps 1 and 2 for the reduced transportation table until all the requirements are satisfied

Determine the initial basic feasible solution using Row Minima Method

1.

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	80	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
	(40)	(80)	(70)	(20)	

F_3					18
	5	8	7	7	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	8 (30)	(40)	(60)	1
F_3	(40)	(80)	(70)	(20)	18
	5	X	7	7	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	8 (30)	1 (40)	(60)	X
F_3	(40)	(80)	(70)	(20)	18
	5	X	6	7	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	8 (30)	1 (40)	(60)	X
F_3	5 (40)	(80)	6 (70)	7 (20)	X
	X	X	X	X	

Initial Basic Feasible Solution
$x_{14} = 7$, $x_{22} = 8$, $x_{23} = 1$, $x_{31} = 5$, $x_{33} = 6$, $x_{34} = 7$
The transportation cost is 7 (10) + 8 (30) + 1 (40) + 5 (40) + 6 (70) + 7 (20) = Rs. 1110

2.

	A	B	C	Availability
I	50	30	220	1
II	90	45	170	4
III	250	200	50	4
Requirement	4	2	3	

Solution

	A	B	C	Availability
I		1 (30)		1 0
II	3 (90)	1 (45)		4 3 0
III	1 (250)		3 (50)	4 1 0
Requirement	4 1 0	2 1 0	3 0	

Initial Basic Feasible Solution
$x_{12} = 1$, $x_{21} = 3$, $x_{22} = 1$, $x_{31} = 1$, $x_{33} = 3$
The transportation cost is 1 (30) + 3 (90) + 1 (45) + 1 (250) + 3 (50) = Rs. 745

Column Minima Method

Step 1
Determine the smallest cost in the first column of the transportation table. Allocate $x_{i1} = \min(a_i, b_1)$ in the cell (i, 1).

Step 2
- If $x_{i1} = b_1$, cross out the first column of the table and move towards right to the second column
- If $x_{i1} = a_i$, cross out the i^{th} row of the table and reconsider the first column with the remaining demand.
- If $x_{i1} = b_1 = a_i$, cross out the i^{th} row and make the second allocation $x_{k1} = 0$ in the cell (k, 1) with c_{k1} being the new minimum cost in the first column, cross out the column and move towards right to the second column.

Step 3

Repeat steps 1 and 2 for the reduced transportation table until all the requirements are satisfied.

Use Column Minima method to determine an initial basic feasible solution

1.

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	80	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	
F_1	5 (19)	(30)	(50)	(10)	2
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	(80)	(70)	(20)	18
	X	8	7	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	(80)	(70)	(20)	18
	X	6	7	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	6 (30)	(40)	(60)	3
F_3	(40)	(80)	(70)	(20)	18
	X	X	7	14	

W_1	W_2	W_3	W_4

F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	6 (30)	3 (40)	(60)	X
	(40)	(80)F[3]	(70)	(20)	
		18			
	X	X	4	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	6 (30)	3 (40)	(60)	X
F_3	(40)	(80)	4 (70)	(20)	14
	X	X	X	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	6 (30)	3 (40)	(60)	X
F_3	(40)	(80)	4 (70)	14 (20)	X
	X	X	X	X	

Initial Basic Feasible Solution

$x_{11} = 5$, $x_{12} = 2$, $x_{22} = 6$, $x_{23} = 3$, $x_{33} = 4$, $x_{34} = 14$

The transportation cost is 5 (19) + 2 (30) + 6 (30) + 3 (40) + 4 (70) + 14 (20) = Rs. 1015

2.

	D_1	D_2	D_3	D_4	Availability
S_1	11	13	17	14	250
S_2	16	18	14	10	300
S_3	21	24	13	10	400
Requirement	200	225	275	250	

Solution

	D1	D2	D3	D4	
S1	200 (11)	50 (13)			250 50 0
S2		175 (18)		125 (10)	300 125 0
S3			275 (13)	125 (10)	400 125 0
	200 0	225 175 0	275 0	250 0	

Initial Basic Feasible Solution

$x_{11} = 200$, $x_{12} = 50$, $x_{22} = 175$, $x_{24} = 125$, $x_{33} = 275$, $x_{34} = 125$

The transportation cost is

200 (11) + 50 (13) + 175 (18) + 125 (10) + 275 (13) + 125 (10) = Rs. 12075

Lowest Cost Entry Method (Matrix Minima Method)

Step 1

Determine the smallest cost in the cost matrix of the transportation table. Allocate $x_{ij} = \min(a_i, b_j)$ in the cell (i, j)

Step 2

- If $x_{ij} = a_i$, cross out the i^{th} row of the table and decrease b_j by a_i. Go to step 3.
- If $x_{ij} = b_j$, cross out the j^{th} column of the table and decrease a_i by b_j. Go to step 3.
- If $x_{ij} = a_i = b_j$, cross out the i^{th} row or j^{th} column but not both.

Step 3

Repeat steps 1 and 2 for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

Find the initial basic feasible solution using Matrix Minima method

1.

	W1	W2	W3	W4	Availability
F1	19	30	50	10	7
F2	70	30	40	60	9
F3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	(10)	7
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	(20)	10
	5	X	7	14	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	(20)	10
	5	X	7	7	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	7 (20)	3
	5	X	7	X	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	3 (40)	8 (8)	(70)	7 (20)	X
	2	X	7	X	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	2 (70)	(30)	7 (40)	(60)	X
F_3	3 (40)	8 (8)	(70)	7 (20)	X
	X	X	X	X	

Initial Basic Feasible Solution
$x_{14} = 7$, $x_{21} = 2$, $x_{23} = 7$, $x_{31} = 3$, $x_{32} = 8$, $x_{34} = 7$
The transportation cost is 7 (10) + 2 (70) + 7 (40) + 3 (40) + 8 (8) + 7 (20) = Rs. 814

2.

	To					Availability
From	2	11	10	3	7	4
	1	4	7	2	1	8
	3	9	4	8	12	9
Requirement	3	3	4	5	6	

Solution
To

From				4 (3)		4 0
	3 (1)				5 (1)	8 5 0
		3 (9)	4 (4)	1 (8)	1 (12)	9 5 4 1 0
	3 0	3 0	4 0	5 1 0	6 1 0	

Initial Basic Feasible Solution
$x_{14} = 4$, $x_{21} = 3$, $x_{25} = 5$, $x_{32} = 3$, $x_{33} = 4$, $x_{34} = 1$, $x_{35} = 1$
The transportation cost is 4 (3) + 3 (1) + 5(1) + 3 (9) + 4 (4) + 1 (8) + 1 (12) = Rs. 78

Vogel's Approximation Method (Unit Cost Penalty Method)

Step1

For each row of the table, identify the **smallest** and the **next to smallest cost**. Determine the difference between them for each row. These are called **penalties**. Put them aside by enclosing them in the parenthesis against the respective rows. Similarly compute penalties for each column.

Step 2

Identify the row or column with the largest penalty. If a tie occurs then use an arbitrary choice. Let the largest penalty corresponding to the i^{th} row have the cost c_{ij}. Allocate the largest possible amount $x_{ij} = \min(a_i, b_j)$ in the cell (i, j) and cross out either i^{th} row or j^{th} column in the usual manner.

Step 3

Again compute the row and column penalties for the reduced table and then go to step 2. Repeat the procedure until all the requirements are satisfied.

Find the initial basic feasible solution using vogel's approximation method

1.

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	19	30	50	10	7	19-10=9
F_2	70	30	40	60	9	40-30=10
F_3	40	8	70	20	18	20-8=12
Requirement	5	8	7	14		
Penalty	40-19=21	30-8=22	50-40=10	20-10=10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	(19)	(30)	(50)	(10)	7	9
F_2	(70)	(30)	(40)	(60)	9	10
F_3	(40)	**8**(8)	(70)	(20)	18/10	12
Requirement	5	8/0	7	14		
Penalty	21	22	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	(10)	7/2	9
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	(20)	18/10	20
Requirement	5/0	X	7	14		
Penalty	21	X	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	(10)	7/2	40
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	**10**(20)	18/10/0	50
Requirement	X	X	7	14/4		
Penalty	X	X	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	7/2/0	40
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	7	14/4/2		
Penalty	X	X	10	50		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	X	X
F_2	(70)	(30)	**7**(40)	**2**(60)	X	X
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	X	X		
Penalty	X	X	X	X		

Initial Basic Feasible Solution

$x_{11} = 5$, $x_{14} = 2$, $x_{23} = 7$, $x_{24} = 2$, $x_{32} = 8$, $x_{34} = 10$

The transportation cost is 5 (19) + 2 (10) + 7 (40) + 2 (60) + 8 (8) + 10 (20) = Rs. 779

2.

		Stores				Availability
		I	II	III	IV	
	A	21	16	115	13	
Warehouse	B	17	18	14	23	13
	C	32	27	18	41	19
Requirement		6	10	12	15	

Solution

		Stores I	II	III	IV	Availability	Penalty
Warehouse	A	(21)	(16)	(15)	(13)	11	2
	B	(17)	(18)	(14)	(23)	13	3
	C	(32)	(27)	(18)	(41)	19	9
Requirement		6	10	12	15		
Penalty		4	2	1	10		

		Stores I	II	III	IV	Availability	Penalty
Warehouse	A	(21)	(16)	(15)	**11**(13)	11/0	2
	B	(17)	(18)	(14)	(23)	13	3
	C	(32)	(27)	(18)	(41)	19	9
Requirement		6	10	12	15/4		
Penalty		4	2	1	10		

		Stores I	II	III	IV	Availability	Penalty
Warehouse	A	(21)	(16)	(15)	**11**(13)	X	X
	B	(17)	(18)	(14)	**4**(23)	13/9	3
	C	(32)	(27)	(18)	(41)	19	9
Requirement		6	10	12	15/4/0		
Penalty		15	9	4	18		

		Stores I	II	III	IV	Availability	Penalty
Warehouse	A	(21)	(16)	(15)	**11**(13)	X	X
	B	**6**(17)	(18)	(14)	**4**(23)	13/9/3	3
	C	(32)	(27)	(18)	(41)	19	9
Requirement		6/0	10	12	X		
Penalty		15	9	4	X		

		Stores					
		I	II	III	IV	Availability	Penalty
Warehouse	A	(21)	(16)	(15)	**11**(13)	X	X
	B	**6**(17)	**3**(18)	(14)	**4**(23)	13/9/3/0	4
	C	(32)	(27)	(18)	(41)	19	9
Requirement		X	10/7	12	X		
Penalty		X	9	4	X		

		Stores					
		I	II	III	IV	Availability	Penalty
Warehouse	A	X(21)	(16) X	X(15)	**11**(13) X	X	X
	B	**6**(17)	**3**(18)	(14)	**4**(23)		
	C	(32)	**7**(27)	**12**(18)	(41)		
Requirement		X	X	X	X		
Penalty		X	X	X	X		

Initial Basic Feasible Solution

$x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$, $x_{33} = 12$

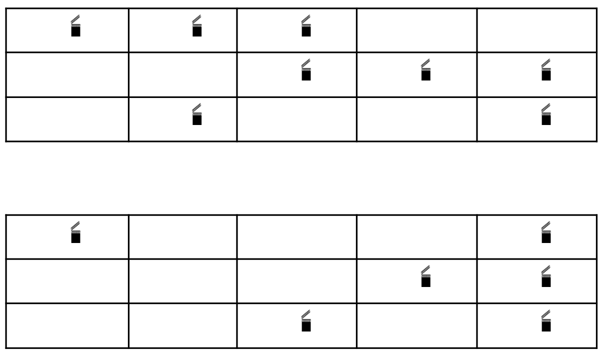
The transportation cost is 11 (13) + 6 (17) + 3 (18) + 4 (23) + 7 (27) + 12 (18) = Rs. 796

Examining the Initial Basic Feasible Solution for Non-Degeneracy

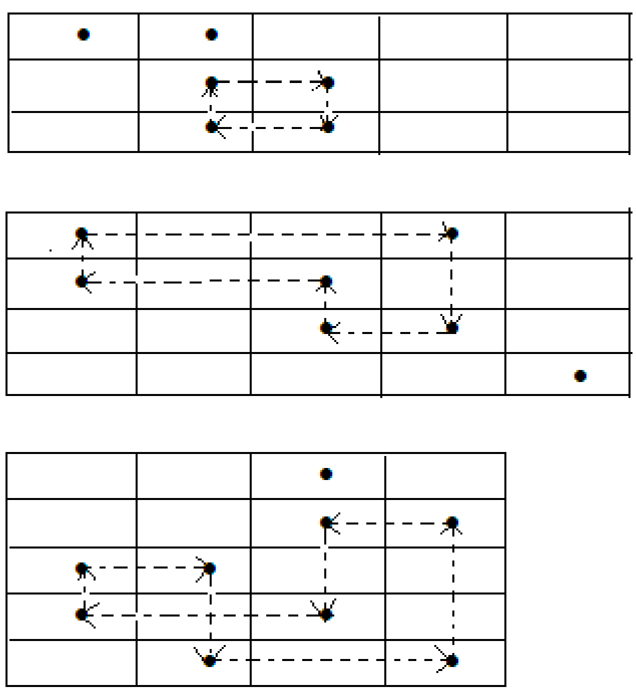
Examine the initial basic feasible solution for non-degeneracy. If it is said to be non-degenerate then it has the following two properties

- Initial basic feasible solution must contain exactly m + n – 1 number of individual allocations.
- These allocations must be in independent positions

Independent Positions



Non-Independent Positions



3.2 Transportation Algorithm for Minimization Problem (MODI Method)

Step 1

Construct the transportation table entering the origin capacities a_i, the destination requirement b_j
and the cost c_{ij}

Step 2

Find an initial basic feasible solution by vogel's method or by any of the given method.

Step 3
For all the basic variables x_{ij}, solve the system of equations $u_i + v_j = c_{ij}$, for all i, j for which cell (i, j) is in the basis, starting initially with some $u_i = 0$, calculate the values of u_i and v_j on the transportation table

Step 4
Compute the cost differences $d_{ij} = c_{ij} – (u_i + v_j)$ for all the non-basic cells

Step 5
Apply optimality test by examining the sign of each d_{ij}

- If all $d_{ij} \geq 0$, the current basic feasible solution is optimal
- If at least one $d_{ij} < 0$, select the variable x_{rs} (most negative) to enter the basis.
- Solution under test is not optimal if any d_{ij} is negative and further improvement is required by repeating the above process.

Step 6
Let the variable x_{rs} enter the basis. Allocate an unknown quantity Θ to the cell (r, s). Then construct a loop that starts and ends at the cell (r, s) and connects some of the basic cells. The amount Θ is added to and subtracted from the transition cells of the loop in such a manner that the availabilities and requirements remain satisfied.

Step 7
Assign the largest possible value to the Θ in such a way that the value of at least one basic variable becomes zero and the other basic variables remain non-negative. The basic cell whose allocation has been made zero will leave the basis.

Step 8
Now, return to step 3 and repeat the process until an optimal solution is obtained.

Worked Examples

Example 1
Find an optimal solution

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

1. **Applying vogel's approximation method for finding the initial basic feasible solution**

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	X	X
F_2	(70)	(30)	**7**(40)	**2**(60)	X	X
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	X	X		
Penalty	X	X	X	X		

Minimum transportation cost is 5 (19) + 2 (10) + 7 (40) + 2 (60) + 8 (8) + 10 (20) = Rs. 779

2. **Check for Non-degeneracy**
The initial basic feasible solution has m + n – 1 i.e. 3 + 4 – 1 = 6 allocations in independent positions. Hence optimality test is satisfied.

3. **Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$**

					u_i
	■ (19)			■ (10)	u_1= -10
			■ (40)	■ (60)	u_2 = 40
		■ (8)		■ (20)	u_3 = 0
v_j	$v_1 = 29$	$v_2 = 8$	$v_3 = 0$	$v_4 = 20$	

Assign a 'u' value to zero. (Convenient rule is to select the u_i, which has the largest number of allocations in its row)
Let $u_3 = 0$, then
$u_3 + v_4 = 20$ which implies $0 + v_4 = 20$, so $v_4 = 20$
$u_2 + v_4 = 60$ which implies $u_2 + 20 = 60$, so $u_2 = 40$
$u_1 + v_4 = 10$ which implies $u_1 + 20 = 10$, so $u_1 = -10$
$u_2 + v_3 = 40$ which implies $40 + v_3 = 40$, so $v_3 = 0$
$u_3 + v_2 = 8$ which implies $0 + v_2 = 8$, so $v_2 = 8$
$u_1 + v_1 = 19$ which implies $-10 + v_1 = 19$, so $v_1 = 29$

4. **Calculation of cost differences for non basic cells $d_{ij} = c_{ij} – (u_i + v_j)$**

c_{ij}

■	(30)	(50)	■
(70)	(30)	■	■
(40)	■	(70)	■

$u_i + v_j$

■	-2	-10	■
69	48	■	■
29	■	0	■

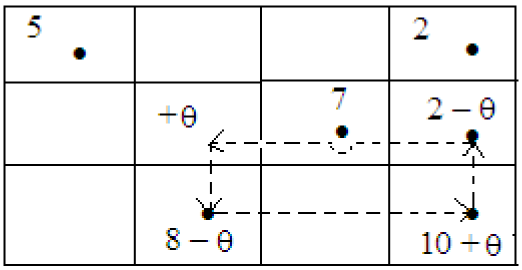
$d_{ij} = c_{ij} – (u_i + v_j)$

■	32	60	■
1	**-18**	■	■
11	■	70	■

5. **Optimality test**

$d_{ij} < 0$ i.e. $d_{22} = -18$
so x_{22} is entering the basis

6. **Construction of loop and allocation of unknown quantity Θ**



We allocate Θ to the cell (2, 2). Reallocation is done by transferring the maximum possible amount Θ in the marked cell. The value of Θ is obtained by equating to zero to the corners of the closed loop. i.e. min(8-Θ, 2-Θ) = 0 which gives Θ = 2. Therefore x_{24} is outgoing as it becomes zero.

5 (19)			2 (10)
	2 (30)	7 (40)	
	6 (8)		12 (20)

Minimum transportation cost is 5 (19) + 2 (10) + 2 (30) + 7 (40) + 6 (8) + 12 (20) = Rs. 743

7. **Improved Solution**

				u_i
■ (19)			■ (10)	$u_1 = -10$
	■ (30)	■ (40)		$u_2 = 22$
	■ (8)		■ (20)	$u_3 = 0$
$v_1 = 29$	$v_2 = 8$	$v_3 = 18$	$v_4 = 20$	v_j

c_{ij}

■	(30)	(50)	■
(70)	■	■	(60)
(40)	■	(70)	■

$u_i + v_j$

■	-2	8	■
51	■	■	42
29	■	18	■

$d_{ij} = c_{ij} - (u_i + v_j)$

■	32	42	■
19	■	■	18
11	■	52	■

Since $d_{ij} > 0$, an optimal solution is obtained with minimal cost Rs.743

Example 2

Solve by lowest cost entry method and obtain an optimal solution for the following problem

				Available
From	50	30	220	1
	90	45	170	3
	250	200	50	4
Required	4	2	2	

Solution

By lowest cost entry method

				Available
From		1(30)		1/0
	2(90)	1(45)		3/2/0
	2(250)		2(50)	4/2/0
Required	4/2/2	2/1/0	2/0	

Minimum transportation cost is 1 (30) + 2 (90) + 1 (45) + 2 (250) + 2 (50) = Rs. 855

Check for Non-degeneracy

The initial basic feasible solution has m + n – 1 i.e. 3 + 3 – 1 = 5 allocations in independent positions. Hence optimality test is satisfied.

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
		✓ (30)		$u_1 = -15$
	✓ (90)	✓ (45)		$u_2 = 0$
	✓ (250)		✓ (50)	$u_3 = 160$
v_j	$v_1 = 90$	$v_2 = 45$	$v_3 = -110$	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

50	✓	220
✓	✓	170
✓	200	✓

$u_i + v_j$

75	✓	-125
✓	✓	-110
✓	205	✓

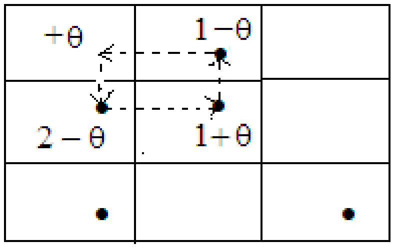
$d_{ij} = c_{ij} – (u_i + v_j)$

-25	✓	345
✓	✓	280
✓	-5	✓

Optimality test

$d_{ij} < 0$ i.e. $d_{11} = -25$ is most negative
So x_{11} is entering the basis

Construction of loop and allocation of unknown quantity Θ



min(2-Θ, 1-Θ) = 0 which gives Θ = 1. Therefore x_{12} is outgoing as it becomes zero.

1(50)		
1(90)	2(45)	
2(250)		2(50)

Minimum transportation cost is 1 (50) + 1 (90) + 2 (45) + 2 (250) + 2 (50) = Rs. 830

II Iteration
Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
	[illegible] (50)			$u_1 = -40$
	[illegible] (90)	[illegible] (45)		$u_2 = 0$
	[illegible] (250)		[illegible] (50)	$u_3 = 160$
v_j	$v_1 = 90$	$v_2 = 45$	$v_3 = -110$	

Calculation of $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

[illegible]	30	220

$u_i + v_j$

[illegible]	5	-150

[illegible]	[illegible]	170
[illegible]	200	[illegible]

[illegible]	[illegible]	-110
[illegible]	205	[illegible]

$d_{ij} = c_{ij} – (u_i + v_j)$

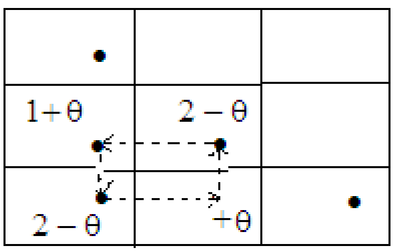
[illegible]	25	370
[illegible]	[illegible]	280
[illegible]	-5	[illegible]

Optimality test

$d_{ij} < 0$ i.e. $d_{32} = -5$

So x_{32} is entering the basis

Construction of loop and allocation of unknown quantity ϴ



$2 – ϴ = 0$ which gives $ϴ = 2$. Therefore x_{22} and x_{31} is outgoing as it becomes zero.

1(50)		
3(90)	0(45)	
	2(200)	2(50)

Minimum transportation cost is 1 (50) + 3 (90) + 2 (200) + 2 (50) = Rs. 820

III Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
	[illegible] (50)			$u_1 = -40$
	[illegible] (90)	[illegible] (45)		$u_2 = 0$
		[illegible] (200)	[illegible] (50)	$u_3 = 155$
v_j	$v_1 = 90$	$v_2 = 45$	$v_3 = -105$	

Calculation of $d_{ij} = c_{ij} – (u_i + v_j)$

$\mathbf{c_{ij}}$

■	30	220
■	■	170
250	■	■

$\mathbf{u_i + v_j}$

■	5	-145
■	■	-105
245	■	■

$\mathbf{d_{ij} = c_{ij} - (u_i + v_j)}$

■	25	365
■	■	275
5	■	■

Since $d_{ij} > 0$, an optimal solution is obtained with minimal cost Rs.820

Example 3

Is $x_{13} = 50$, $x_{14} = 20$, $x_{21} = 55$, $x_{31} = 30$, $x_{32} = 35$, $x_{34} = 25$ an optimal solution to the transportation problem.

					Available
	6	1	9	3	70
From	11	5	2	8	55
	10	12	4	7	90
Required	85	35	50	45	

Solution

					Available
			50(9)	20(3)	X
From	55(11)				X
	30(10)	35(12)		25(7)	X
Required	X	X	X	X	

Minimum transportation cost is 50 (9) + 20 (3) + 55 (11) + 30 (10) + 35 (12) + 25 (7) = Rs. 2010

Check for Non-degeneracy

The initial basic feasible solution has m + n – 1 i.e. 3 + 4 – 1 = 6 allocations in independent positions. Hence optimality test is satisfied.

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
		✓ (9)	✓ (3)	$u_1 = -4$
✓ (11)				$u_2 = 1$
✓ (10)	✓ (12)		✓ (7)	$u_3 = 0$
v_j: $v_1 = 10$	$v_2 = 12$	$v_3 = 13$	$v_4 = 7$	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

6	1	✓	✓
✓	5	2	8
✓	✓	4	✓

$u_i + v_j$

6	8	✓	✓
✓	13	14	8
✓	✓	13	✓

$d_{ij} = c_{ij} – (u_i + v_j)$

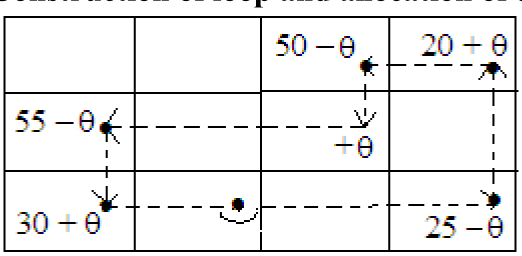
0	-7	✓	✓
✓	-8	-12	0
✓	✓	-9	✓

Optimality test

$d_{ij} < 0$ i.e. $d_{23} = -12$ is most negative

So x_{23} is entering the basis

Construction of loop and allocation of unknown quantity Θ



min(50-Θ, 55-Θ, 25-Θ) = 25 which gives Θ = 25. Therefore x_{34} is outgoing as it becomes zero.

		25(9)	45(3)
30(11)		25(2)	
55(10)	35(12)		

Minimum transportation cost is 25 (9) + 45 (3) + 30 (11) + 25 (2) + 55 (10) + 35 (12) = Rs. 1710

II iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

					u_i
			■ (9)	■ (3)	$u_1 = 8$
	■ (11)		■ (2)		$u_2 = 1$
	■ (10)	■ (12)			$u_3 = 0$
v_j	$v_1 = 10$	$v_2 = 12$	$v_3 = 1$	$v_4 = -5$	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} - (u_i + v_j)$

c_{ij}

6	1	■	■
■	5	■	8
■	■	4	7

$u_i + v_j$

18	20	■	■
■	13	■	-4
■	■	1	-5

$d_{ij} = c_{ij} - (u_i + v_j)$

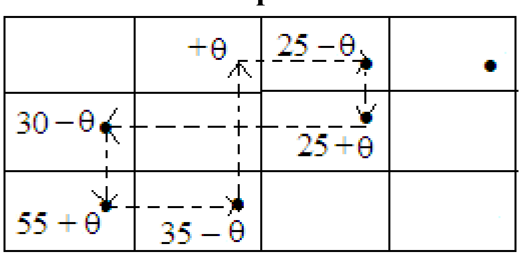
-12	-19	■	■
■	-8	■	12
■	■	3	12

Optimality test

$d_{ij} < 0$ i.e. $d_{12} = -19$ is most negative

So x_{12} is entering the basis

Construction of loop and allocation of unknown quantity Θ



min(25-Θ, 30-Θ, 35-Θ) = 25 which gives Θ = 25. Therefore x_{13} is outgoing as it becomes zero.

	25(1)		45(3)
5(11)		50(2)	
80(10)	10(12)		

Minimum transportation cost is 25 (1) + 45 (3) + 5 (11) + 50 (2) + 80 (10) + 10 (12) = Rs. 1235

III Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
	🎬 (1)		🎬 (3)	$u_1 = -11$
🎬 (11)		🎬 (2)		$u_2 = 1$
🎬 (10)	🎬 (12)			$u_3 = 0$
$v_1 = 10$	$v_2 = 12$	$v_3 = 1$	$v_4 = 14$	

v_j

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} - (u_i + v_j)$

c_{ij}

6	🎬	9	🎬
🎬	5	🎬	8
🎬	🎬	4	7

$u_i + v_j$

-1	🎬	-10	🎬
🎬	13	🎬	15
🎬	🎬	1	14

$d_{ij} = c_{ij} - (u_i + v_j)$

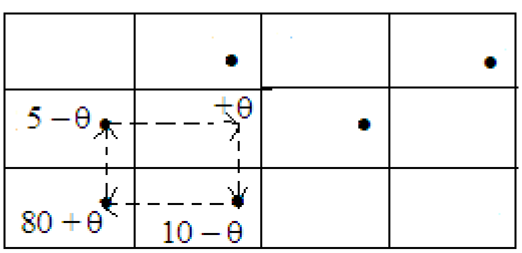
7	🎬	19	🎬
🎬	-8	🎬	-7
🎬	🎬	3	-7

Optimality test

$d_{ij} < 0$ i.e. $d_{22} = -8$ is most negative

So x_{22} is entering the basis

Construction of loop and allocation of unknown quantity Θ



min(5-Θ, 10-Θ) = 5 which gives Θ = 5. Therefore x_{21} is outgoing as it becomes zero.

	25(1)		45(3)
	5(5)	50(2)	
85(10)	5(12)		

Minimum transportation cost is 25 (1) + 45 (3) + 5 (5) + 50 (2) + 85 (10) + 5 (12) = Rs. 1195

IV Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

					u_i
		✓ (1)		✓ (3)	$u_1 = -11$
		✓ (5)	✓ (2)		$u_2 = -7$
	✓ (10)	✓ (12)			$u_3 = 0$
v_j	$v_1 = 10$	$v_2 = 12$	$v_3 = 9$	$v_4 = 14$	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} - (u_i + v_j)$

c_{ij}

6	✓	9	✓
11	✓	✓	8
✓	✓	4	7

$u_i + v_j$

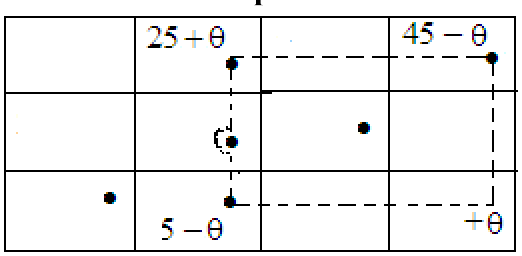
-1	✓	-2	✓
3	✓	✓	7
✓	✓	9	14

$d_{ij} = c_{ij} - (u_i + v_j)$

7	✓	11	✓
8	✓	✓	1
✓	✓	-5	-7

Optimality test
$d_{ij} < 0$ i.e. $d_{34} = -7$ is most negative
So x_{34} is entering the basis

Construction of loop and allocation of unknown quantity ϴ



min(5-ϴ, 45-ϴ) = 5 which gives ϴ = 5. Therefore x_{32} is outgoing as it becomes zero.

	30(1)		40(3)
	5(5)	50(2)	
85(10)			5(7)

Minimum transportation cost is 30 (1) + 40 (3) + 5 (5) + 50 (2) + 85 (10) + 5 (7) = Rs. 1160

V Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

					u_i
		■ (1)		■ (3)	$u_1 = -4$
		■ (5)	■ (2)		$u_2 = 0$
	■ (10)			■ (7)	$u_3 = 0$
v_j	$v_1 = 10$	$v_2 = 5$	$v_3 = 2$	$v_4 = 7$	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

6	■	9	■
11	■	■	8
■	12	4	■

$u_i + v_j$

6	■	-2	■
10	■	■	7
■	5	2	■

$d_{ij} = c_{ij} – (u_i + v_j)$

0	■	11	■
1	■	■	1
■	7	2	■

Since $d_{ij} > 0$, an optimal solution is obtained with minimal cost Rs.1160. Further more d_{11} = 0 which indicates that alternative optimal solution also exists.

Introduction to Assignment Problem

In assignment problems, the objective is to assign a number of jobs to the equal number of persons at a minimum cost of maximum profit.

Suppose there are 'n' jobs to be performed and 'n' persons are available for doing these jobs. Assume each person can do each job at a time with a varying degree of efficiency. Let c_{ij} be the cost of i^{th} person assigned to j^{th} job. Then the problem is to find an assignment so that the total cost for performing all jobs is minimum. Such problems are known as **assignment problems.**

These problems may consist of assigning men to offices, classes to the rooms or problems to the research team etc.

Mathematical formulation

Cost matrix: $c_{ij}=$

$$\begin{matrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ . \\ . \\ . \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nn} \end{matrix}$$

$$\text{Minimize cost : } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} \qquad i = 1, 2, .., n \quad j = 1, 2, .., n$$

Subject to restrictions of the form

$$x_{ij} = \begin{cases} 1 \text{ if ith person is assigned jth job} \\ 0 \text{ if not} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{(one job is done by the ith person, } i = 1,2,\dots n)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{(only one person should be assigned the jth job, } j = 1,2,\dots n)$$

Where x_{ij} denotes that j^{th} job is to be assigned to the i^{th} person.

This special structure of assignment problem allows a more convenient method of solution in comparison to simplex method.

Algorithm for Assignment Problem (Hungarian Method)

Step 1

Subtract the minimum of each row of the effectiveness matrix, from all the elements of the respective rows (Row reduced matrix).

Step 2

Further modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus first modified matrix is obtained.

Step 3

Draw the minimum number of horizontal and vertical lines to cover all the zeroes in the resulting matrix. Let the minimum number of lines be N. Now there may be two possibilities

- If N = n, the number of rows (columns) of the given matrix then an optimal assignment can be made. So make the zero assignment to get the required solution.
- If N < n then proceed to step 4

Step 4

Determine the smallest element in the matrix, not covered by N lines. Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus the second modified matrix is obtained.

Step 5

Repeat step 3 and step 4 until minimum number of lines become equal to number of rows (columns) of the given matrix i.e. N = n.

Step 6

To make zero assignment - examine the rows successively until a row-wise exactly single zero is found; mark this zero by '□' to make the assignment. Then, mark a 'X' over all zeroes if lying in the column of the marked zero, showing that they cannot be considered for further assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for the columns also.

Step 7

Repeat the step 6 successively until one of the following situations arise

- If no unmarked zero is left, then process ends
- If there lies more than one of the unmarked zeroes in any column or row, then mark '□' one of the unmarked zeroes arbitrarily and mark a cross in the cells of remaining zeroes in its row and column. Repeat the process until no unmarked zero is left in the matrix.

Step 8

Exactly one marked zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked zeroes will give the optimal assignment.

Worked Examples

Example 1

A department head has four subordinates and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man-hours?

	Subordinates			
Tasks	I	II	III	IV
A	8	26	17	11

B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

Solution

Row Reduced Matrix

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

I Modified Matrix

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

N = 4, n = 4
Since N = n, we move on to zero assignment

Zero assignment

[0]	14	9	3
9	20	[0]	22
23	[0]	3	~~0~~
9	12	14	[0]

Optimal assignment	A – I	B – III	C – II	D – IV
Man-hours	8	4	19	10

Total man-hours = 8 + 4 + 19 + 10 = 41 hours

Example 2

A car hire company has one car at each of five depots a, b, c, d and e. a customer requires a car in each town namely A, B, C, D and E. Distance (kms) between depots (origins) and towns (destinations) are given in the following distance matrix

	a	b	c	d	e
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

Solution

Row Reduced Matrix

30	0	45	60	70
15	0	10	40	55
30	0	45	60	75
0	0	30	30	60
20	0	35	45	70

I Modified Matrix

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

N < n i.e. 3 < 5, so move to next modified matrix

II Modified Matrix

15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

N = 5, n = 5
Since N = n, we move on to zero assignment
Zero assignment

15	0 (crossed)	20	15	[0]
15	15	[0]	10	0 (crossed)
15	[0]	20	15	5
[0]	15	20	0 (crossed)	5
5	0 (crossed)	10	[0]	0 (crossed)

Route	A – e	B – c	C – b	D – a	E – d
Distance	200	130	110	50	80

Minimum distance travelled = 200 + 130 + 110 + 50 + 80 = 570 kms

Example 3
Solve the assignment problem whose effectiveness matrix is given in the table

1 2 3 4

A	49	60	45	61
B	55	63	45	69
C	52	62	49	68
D	55	64	48	66

Solution

Row-Reduced Matrix

4	15	0	16
10	18	0	24
3	13	0	19
7	16	0	18

I Modified Matrix

1	2	0	0
7	5	0	8
0	0	0	3
4	3	0	2

N < n i.e 3 < 4, so II modified matrix

II Modified Matrix

1	2	2	0
5	3	0	6
0	0	2	3
2	1	0	0

N < n i.e 3 < 4

III Modified matrix

0	1	2	0
4	2	0	6
0	0	3	4
1	0	0	0

Since N = n, we move on to zero assignment

Zero assignment

Multiple optimal assignments exists

Solution - I

[0]	1	2	~~0~~
4	2	[0]	6
~~0~~	[0]	3	4
1	~~0~~	~~0~~	[0]

Optimal assignment	A – 1	B – 3	C – 2	D - 4
Value	49	45	62	66

Total cost = 49 + 45 + 62 + 66 = 222 units

Solution – II

~~0~~	1	2	[0]
4	2	[0]	6
[0]	~~0~~	3	4
1	[0]	~~0~~	~~0~~

Optimal assignment	A – 4	B – 3	C – 1	D – 2
Value	61	45	52	64

Minimum cost = 61 + 45 + 52 + 64 = 222 units

Example 4

Certain equipment needs 5 repair jobs which have to be assigned to 5 machines. The estimated time (in hours) that a mechanic requires to complete the repair job is given in the table.

Assuming that each mechanic can be assigned only one job, determine the minimum time assignment.

	J1	J2	J3	J4	J5
M1	7	5	9	8	11
M2	9	12	7	11	10
M3	8	5	4	6	9
M4	7	3	6	9	5
M5	4	6	7	5	11

Solution

Row Reduced Matrix

2	0	4	3	6
2	5	0	4	3
4	1	0	2	5
4	0	3	6	2
0	2	3	1	7

I Modified Matrix

2	0	4	2	4
2	5	0	3	1
4	1	0	1	3
4	0	3	5	0
0	2	3	0	5

N < n

II Modified Matrix

1	0	4	1	3
1	5	0	2	0
3	1	0	0	2
4	1	4	5	0
0	3	4	0	5

N = n

Zero assignment

1	[0]	4	1	3
1	5	[0]	2	~~0~~
3	1	~~0~~	[0]	2
4	1	4	5	[0]
[0]	3	4	~~0~~	5

Optimal assignment	M1 – J2	M2 – J3	M3 – J4	M4 – J5	M5 – J1
Hours	5	7	6	5	4

Minimum time = 5 + 7 + 6 + 5 + 4 = 27 hours

Unbalanced Assignment Problems

If the number of rows and columns are not equal then such type of problems are called as unbalanced assignment problems.

Example 1

A company has 4 machines on which to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table

		Machines			
		W	X	Y	Z
Jobs	A	18	24	28	32
	B	8	13	17	19
	C	10	15	19	22

Solution

18	24	28	32
8	13	17	19
10	15	19	22
0	0	0	0

Row Reduced matrix

0	6	10	14
0	5	9	11
0	5	9	12
0	0	0	0

I Modified Matrix

0	6	10	14
0	5	9	11
0	5	9	12
0	0	0	0

N < n i.e. 2 < 4

II Modified Matrix

0	1	5	9
0	0	4	6
0	0	4	7
5	0	0	0

N < n i.e. 3 < 4

III Modified Matrix

0	1	1	5
0	0	0	2
0	0	0	3
9	4	0	0

N = n

Zero assignment

Multiple assignments exists

Solution -I

[0]	1	1	5
~~0~~	[0]	~~0~~	2
~~0~~	~~0~~	[0]	3
9	4	~~0~~	[0]

Optimal assignment	W – A	X – B	Y – C
Cost	18	13	19

Minimum cost = 18 + 13 + 19 = Rs 50

Solution -II

[0]	1	1	5
~~0~~	~~0~~	[0]	2
~~0~~	[0]	~~0~~	3
9	4	~~0~~	[0]

Optimal assignment	W – A	X – C	Y – B
Cost	18	17	15

Minimum cost = 18 + 17 + 15 = Rs 50

Example 2

Solve the assignment problem whose effectiveness matrix is given in the table

	R1	R2	R3	R4
C1	9	14	19	15
C2	7	17	20	19
C3	9	18	21	18
C4	10	12	18	19
C5	10	15	21	16

Solution

9	14	19	15	0
7	17	20	19	0
9	18	21	18	0
10	12	18	19	0
10	15	21	16	0

Row Reduced Matrix

9	14	19	15	0
7	17	20	19	0
9	18	21	18	0
10	12	18	19	0

10	15	21	16	0

I Modified Matrix

2	2	1	0	0
0	5	2	4	0
2	6	3	3	0
3	0	0	4	0
3	3	3	1	0

N < n i.e. 4 < 5

II Modified Matrix

1	1	0	0	0
0	5	2	5	1
1	5	2	3	0
3	0	0	5	1
2	2	2	1	0

N < n i.e. 4 < 5

III Modified Matrix

2	1	0	0	1
0	4	1	4	1
1	4	1	2	0
4	0	0	5	2
2	1	1	0	0

N = n

Zero assignment

2	1	[0]	~~0~~	1
[0]	4	1	4	1
1	4	1	2	[0]
4	[0]	~~0~~	5	2
2	1	1	[0]	~~0~~

Optimal assignment	C1 – R3	C2 – R1	C4 – R2	C5 – R4
Units	19	7	12	16

Minimum cost = 19 + 7 + 12 + 16 = 54 units

Maximal Assignment Problem

Example 1

A company has 5 jobs to be done. The following matrix shows the return in terms of rupees on assigning i^{th} (i = 1, 2, 3, 4, 5) machine to the j^{th} job (j = A, B, C, D, E). Assign the five jobs to the five machines so as to maximize the total expected profit.

Jobs

Machines		A	B	C	D	E
	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

Solution

Subtract all the elements from the highest element
Highest element = 14

9	3	4	2	10
12	10	8	11	9
11	2	9	0	8
8	0	10	3	7
7	5	6	2	9

Row Reduced matrix

7	1	2	0	8
4	2	0	3	1
11	2	9	0	8
8	0	10	3	7
5	3	4	0	7

I Modified Matrix

3	1	2	0	7
0	2	0	3	0
7	2	9	0	7
4	0	10	3	6
1	3	4	0	6

N < n i.e. 3 < 5

II Modified Matrix

2	0	1	0	6
0	2	0	4	0
6	1	8	0	6
4	0	10	4	6
0	2	3	0	5

N < n i.e. 4 < 5

III Modified Matrix

1	0	0	0	5
0	3	0	5	0
5	1	7	0	5
3	0	9	4	5
0	3	3	1	5

N = n

Zero assignment

1	~~0~~	[0]	~~0~~	5
~~0~~	3	~~0~~	5	[0]
5	1	7	[0]	5
3	[0]	9	4	5
[0]	3	3	1	5

Optimal assignment 1 – C 2 – E 3 – D 4 – B 5 – A

Maximum profit = 10 + 5 + 14 + 14 + 7 = Rs. 50