Module 2

1.1 Introduction

General Linear Programming Problem (GLPP)

Maximize / Minimize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots\ldots\ldots\ldots\ldots + c_nx_n$

Subject to constraints

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n\ (\leq \text{ or } \geq)\ b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n\ (\leq \text{ or } \geq)\ b_2$

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$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n\ (\leq \text{ or } \geq)\ b_m$

and

$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$

Where constraints may be in the form of any inequality ($\leq$ or $\geq$) or even in the form of an equation (=) and finally satisfy the non-negativity restrictions.

1.2 Steps to convert GLPP to SLPP (Standard LPP)

Step 1 – Write the objective function in the maximization form. If the given objective function is of minimization form then multiply throughout by -1 and write Max $z^{'}$ = Min (-z)

Step 2 – Convert all inequalities as equations.

- If an equality of '$\leq$' appears then by adding a variable called **Slack variable.** We can convert it to an equation. For example $x_1 + 2x_2 \leq 12$, we can write as

  $x_1 + 2x_2 + s_1 = 12.$

- If the constraint is of '$\geq$' type, we subtract a variable called **Surplus variable** and convert it to an equation. For example

  $2x_1 + x_2 \geq 15$

  $2x_1 + x_2 - s_2 = 15$

Step 3 – The right side element of each constraint should be made non-negative

$2x_1 + x_2 - s_2 = -15$

$-2x_1 - x_2 + s_2 = 15$ (That is multiplying throughout by -1)

Step 4 – All variables must have non-negative values.

For example: $x_1 + x_2 \leq 3$

$x_1 > 0$, x_2 is unrestricted in sign

Then x_2 is written as $x_2 = x_2' - x_2''$ where x_2', $x_2 0 \leq ''$

Therefore the inequality takes the form of equation as $x_1 + (x_2' - x_2'') + s_1 = 3$

Using the above steps, we can write the GLPP in the form of SLPP.

<u>Write the Standard LPP (SLPP) of the following</u>

Example 1

Maximize $Z = 3x_1 + x_2$

Subject to

$2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

and $x_1 \geq 0, x_2 \geq 0$

SLPP

Maximize $Z = 3x_1 + x_2$

Subject to

$2x_1 + x_2 + s_1 = 2$

$3x_1 + 4x_2 - s_2 = 12$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$

Example 2

Minimize $Z = 4x_1 + 2x_2$

Subject to

$3x_1 + x_2 \geq 2$

$x_1 + x_2 \geq 21$

$x_1 + 2x_2 \geq 30$

and $x_1 \geq 0, x_2 \geq 0$

SLPP

Maximize $Z4 - = {}' x_1 - 2x_2$

Subject to

$3x_1 + x_2 - s_1 = 2$

$x_1 + x_2 - s_2 = 21$

$x_1 + 2x_2 - s_3 = 30$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

Example 3

Minimize $Z = x_1 + 2x_2 + 3x_3$

Subject to

$2x_1 + 3x_2 + 3x_3 \geq -4$
$3x_1 + 5x_2 + 2x_3 \leq 7$
and $x_1 \geq 0$, $x_2 \geq 0$, x_3 is unrestricted in sign

SLPP

Maximize $Z' = -x_1 - 2x_2 - 3(x_3'' - x_3')$
Subject to
$-2x_1 - 3x_2 - 3(x_3' - x_3'') + s_1 = 4$
$3x_1 + 5x_2 + 2(x_3' - x_3'') + s_2 = 7$
$x_1 \geq 0, x_2 \geq 0, x_3 0 \leq', x_3'' \geq 0, s_1 \geq 0, s_2 \geq 0$

1.3 Some Basic Definitions

Solution of LPP
Any set of variable ($x_1, x_2 \ldots x_n$) which satisfies the given constraint is called solution of LPP.

Basic solution
Is a solution obtained by setting any 'n' variable equal to zero and solving remaining 'm' variables. Such 'm' variables are called **Basic variables** and 'n' variables are called **Non-basic variables**.

Basic feasible solution
A basic solution that is feasible (all basic variables are non negative) is called basic feasible solution. There are two types of basic feasible solution.

1. **Degenerate basic feasible solution**
   If any of the basic variable of a basic feasible solution is zero than it is said to be degenerate basic feasible solution.
2. **Non-degenerate basic feasible solution**
   It is a basic feasible solution which has exactly 'm' positive x_i, where i=1, 2, … m. In other words all 'm' basic variables are positive and remaining 'n' variables are zero.

Optimum basic feasible solution
A basic feasible solution is said to be optimum if it optimizes (max / min) the objective function.

1.4 Introduction to Simplex Method

It was developed by G. Danztig in 1947. The simplex method provides an algorithm (a rule of procedure usually involving repetitive application of a prescribed operation) which is based on the fundamental theorem of linear programming.

The Simplex algorithm is an iterative procedure for solving LP problems in a finite number of steps. It consists of

- Having a trial basic feasible solution to constraint-equations
- Testing whether it is an optimal solution

- Improving the first trial solution by a set of rules and repeating the process till an optimal solution is obtained

Advantages
- Simple to solve the problems
- The solution of LPP of more than two variables can be obtained.

1.5 Computational Procedure of Simplex Method

Consider an example

Maximize $Z = 3x_1 + 2x_2$

Subject to

$x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Step 1 – Write the given GLPP in the form of

SLPP Maximize $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$

Subject to

$x_1 + x_2 + s_1 = 4$

$x_1 - x_2 + s_2 = 2$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$

Step 2 – Present the constraints in the matrix

form $x_1 + x_2 + s_1 = 4$

$x_1 - x_2 + s_2 = 2$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Step 3 – Construct the starting simplex table using the notations

		$C_j \rightarrow$	3	2	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B/X_k
s_1	0	4	1	1	1	0	
s_2	0	2	1	-1	0	1	
	$Z = C_B X_B$		Δ_j				

Step 4 – Calculation of Z and Δ_j and test the basic feasible solution for optimality by the rules given.

$Z = C_B X_B$

$= 0 * 4 + 0 * 2 = 0$

$\Delta_j = Z_j - C_j$
$= C_B X_j - C_j$
$\Delta_1 = C_B X_1 - C_j = 0 * 1 + 0 * 1 - 3 = -3$
$\Delta_2 = C_B X_2 - C_j = 0 * 1 + 0 * -1 - 2 = -2$
$\Delta_3 = C_B X_3 - C_j = 0 * 1 + 0 * 0 - 0 = 0$
$\Delta_4 = C_B X_4 - C_j = 0 * 0 + 0 * 1 - 0 = 0$

Procedure to test the basic feasible solution for optimality by the rules given

Rule 1 – If all $\Delta_j \geq 0$, the solution under the test will be **optimal**. Alternate optimal solution will exist if any non-basic Δ_j is also zero.

Rule 2 – If atleast one Δ_j is negative, the solution is not optimal and then proceeds to improve the solution in the next step.

Rule 3 – If corresponding to any negative Δ_j, all elements of the column X_j are negative or zero, then the solution under test will be **unbounded**.

In this problem it is observed that Δ_1 and Δ_2 are negative. Hence proceed to improve this solution

Step 5 – To improve the basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined.

- **Incoming vector**
  The incoming vector X_k is always selected corresponding to the most negative value of Δ_j. It is indicated by (↑).

- **Outgoing vector**
  The outgoing vector is selected corresponding to the least positive value of minimum ratio. It is indicated by (→).

Step 6 – Mark the key element or pivot element by '□'.The element at the intersection of outgoing vector and incoming vector is the pivot element.

Basic Variables	C_B	X_B	X_1 (X_k)	X_2	S_1	S_2	Min ratio X_B / X_k
		Cj →	3	2	0	0	
s_1	0	4	1	1	1	0	4 / 1 = 4
s_2	0	2	[1]	-1	0	1	2 / 1 = 2 → outgoing
	$Z = C_B X_B = 0$		↑incoming $\Delta_1 = -3$	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$	

- If the number in the marked position is other than unity, divide all the elements of that row by the key element.
- Then subtract appropriate multiples of this new row from the remaining rows, so as to obtain zeroes in the remaining position of the column X_k.

Basic Variables	C_B	X_B	X_1	X_2 (X_k)	S_1	S_2	Min ratio X_B/X_k
s_1	0	2	($R_1=R_1 - R_2$) 0	[2]	1	-1	2 / 2 = 1 ⟶ outgoing
x_1	3	2	1	-1	0	1	2 / -1 = -2 (neglect in case of negative)
	Z=0*2+3*2= 6		$\Delta_1=0$	↑incoming $\Delta_2= -5$	$\Delta_3=0$	$\Delta_4=3$	

Step 7 – Now repeat step 4 through step 6 until an optimal solution is obtained.

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B/X_k
x_2	2	1	($R_1=R_1 / 2$) 0	1	1/2	-1/2	
x_1	3	3	($R_2=R_2 + R_1$) 1	0	1/2	1/2	
	Z = 11		$\Delta_1=0$	$\Delta_2=0$	$\Delta_3=5/2$	$\Delta_4=1/2$	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 11, $x_1 = 3$ and $x_2 = 1$

1.6 Worked Examples

Solve by simplex method

Example 1

Maximize $Z = 80x_1 + 55x_2$

Subject to

$4x_1 + 2x_2 \leq 40$

$2x_1 + 4x_2 \leq 32$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Maximize $Z = 80x_1 + 55x_2 + 0s_1 + 0s_2$
Subject to
$4x_1 + 2x_2 + s_1 = 40$
$2x_1 + 4x_2 + s_2 = 32$
$x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0$

		$C_j \rightarrow$	80	55	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	40	[4]	2	1	0	40 / 4 = 10 → outgoing
s_2	0	32	2	4	0	1	32 / 2 = 16
	$Z = C_B X_B = 0$		↑incoming $\Delta_1 = -80$	$\Delta_2 = -55$	$\Delta_3 = 0$	$\Delta_4 = 0$	
x_1	80	10	$(R_1 = R_1 / 4)$ 1	1/2	1/4	0	10/1/2 = 20
s_2	0	12	$(R_2 = R_2 - 2R_1)$ 0	[3]	-1/2	1	12/3 = 4 → outgoing
	Z = 800		$\Delta_1 = 0$	↑incoming $\Delta_2 = -15$	$\Delta_3 = 40$	$\Delta_4 = 0$	
x_1	80	8	$(R_1 = R_1 - 1/2R_2)$ 1	0	1/3	-1/6	
x_2	55	4	$(R_2 = R_2 / 3)$ 0	1	-1/6	1/3	
	Z = 860		$\Delta_1 = 0$	$\Delta_2 = 0$	$\Delta_3 = 35/2$	$\Delta_4 = 5$	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 860, $x_1 = 8$ and $x_2 = 4$

Example 2
Maximize $Z = 5x_1 + 3x_2$
Subject to
$3x_1 + 5x_2 \leq 15$
$5x_1 + 2x_2 \leq 10$
and $x_1 \geq 0,\ x_2 \geq 0$

Solution

SLPP

Maximize $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2$

Subject to

$3x_1 + 5x_2 + s_1 = 15$

$5x_1 + 2x_2 + s_2 = 10$

$x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0$

		$C_j \rightarrow$	5	3	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	15	3	5	1	0	15 / 3 = 5
s_2	0	10	[5]	2	0	1	10 / 5 = 2 → outgoing
	$Z = C_B X_B = 0$		↑incoming $\Delta_1 = -5$	$\Delta_2 = -3$	$\Delta_3 = 0$	$\Delta_4 = 0$	
s_1	0	9	$(R_1 = R_1 - 3R_2)$ 0	[19/5]	1	-3/5	9/19/5 = 45/19 →
x_1	5	2	$(R_2 = R_2 / 5)$ 1	2/5	0	1/5	2/2/5 = 5
	$Z = 10$		$\Delta_1 = 0$	↑ $\Delta_2 = -1$	$\Delta_3 = 0$	$\Delta_4 = 1$	
x_2	3	45/19	$(R_1 = R_1 / 19/5)$ 0	1	5/19	-3/19	
x_1	5	20/19	$(R_2 = R_2 - 2/5\ R_1)$ 1	0	-2/19	5/19	
	$Z = 235/19$		$\Delta_1 = 0$	$\Delta_2 = 0$	$\Delta_3 = 5/19$	$\Delta_4 = 16/19$	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z = 235/19$, $x_1 = 20/19$ and $x_2 = 45/19$

Example 3

Maximize $Z = 5x_1 + 7x_2$

Subject to

$x_1 + x_2 \leq 4$

$3x_1 - 8x_2 \leq 24$

$10x_1 + 7x_2 \leq 35$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

SLPP

Maximize $Z = 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to

$x_1 + x_2 + s_1 = 4$

$3x_1 - 8x_2 + s_2 = 24$

$10x_1 + 7x_2 + s_3 = 35$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

		$C_j \rightarrow$	5	7	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B/X_k
s_1	0	4	1	[1]	1	0	0	4 /1 = 4→outgoing
s_2	0	24	3	-8	0	1	0	–
s_3	0	35	10	7	0	0	1	35 / 7 = 5
				↑incoming				
	$Z = C_B X_B = 0$		-5	-7	0	0	0	←Δ_j
x_2	7	4	1	1	1	0	0	
			($R_2 = R_2 + 8R_1$)					
s_2	0	56	11	0	8	1	0	
			($R_3 = R_3 - 7R_1$)					
s_3	0	7	3	0	-7	0	1	
	$Z = 28$		2	0	7	0	0	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 28, $x_1 = 0$ and $x_2 = 4$

Example 4

Maximize $Z = 2x - 3y + z$

Subject to

$3x + 6y + z \leq 6$

$4x + 2y + z \leq 4$

$x - y + z \leq 3$

and $x \geq 0, y \geq 0, z \geq 0$

Solution

SLPP

Maximize $Z = 2x - 3y + z + 0s_1 + 0s_2 + 0s_3$

Subject to

$3x + 6y + z + s_1 = 6$

$4x + 2y + z + s_2 = 4$

$x - y + z + s_3 = 3$

$x \geq 0, y \geq 0, z \geq 0\ s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

		$C_j \rightarrow$	2	-3	1	0	0	0	
Basic Variables	C_B	X_B	X	Y	Z	S_1	S_2	S_3	Min ratio X_B / X_k
s_1	0	6	3	6	1	1	0	0	6 / 3 = 2
s_2	0	4	[4]	2	1	0	1	0	4 / 4 =1⟶ outgoing
s_3	0	3	1	-1	1	0	0	1	3 / 1 = 3
	Z = 0		↑incoming -2	3	-1	0	0	0	←Δ_j
s_1	0	3	0	9/2	1/4	1	-3/4	0	3/1/4=12
x	2	1	1	1/2	1/4	0	1/4	0	1/1/4=4
s_3	0	2	0	-3/2	[3/4]	0	-1/4	1	8/3 = 2.6⟶
	Z = 2		0	4	↑incoming 1/2	0	1/2	0	←Δ_j
s_1	0	7/3	0	5	0	1	-2/3	-1/3	
x	2	1/3	1	1	0	0	1/3	-1/3	
z	1	8/3	0	-2	1	0	-1/3	4/3	
	Z = 10/3		0	3	0	0	1/3	2/3	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained.

Therefore the solution is Max Z = 10/3, x = 1/3, y = 0 and z =

8/3

Example 5

Maximize $Z = 3x_1 + 5x_2$
Subject to
$3x_1 + 2x_2 \leq 18$
$x_1 \leq 4$
$x_2 \leq 6$
and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Maximize $Z = 3x_1 + 5x_2 + 0s_1 + 0s_2 + 0s_3$
Subject to
$3x_1 + 2x_2 + s_1 = 18$
$x_1 + s_2 = 4$
$x_2 + s_3 = 6$
$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

		$C_j \rightarrow$	3	5	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k
s_1	0	18	3	2	1	0	0	18 / 2 = 9
s_2	0	4	1	0	0	1	0	4 / 0 = ∞ (neglect)
s_3	0	6	0	[1]	0	0	1	6 / 1 = 6→
				↑				
	Z = 0		-3	-5	0	0	0	←Δ_j
			[ ]	($R_1=R_1-2R_3$)				
s_1	0	6	3	0	1	0	-2	6 / 3 = 2 →
s_2	0	4	1	0	0	1	0	4 / 1 = 4
x_2	5	6	0	1	0	0	1	--
			↑					
	Z = 30		-3	0	0	0	5	←Δ_j
			($R_1=R_1 / 3$)					
x_1	3	2	1	0	1/3	0	-2/3	
			($R_2=R_2 - R_1$)					
s_2	0	2	0	0	-1/3	1	2/3	
x_2	5	6	0	1	0	0	1	
	Z = 36		0	0	1	0	3	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 36, $x_1 = 2$, $x_2 = 6$

Example 6

Minimize $Z = x_1 - 3x_2 + 2x_3$

Subject to

$3x_1 - x_2 + 3x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Solution

SLPP

Min (-Z) = Max $Z' = -x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to

$3x_1 - x_2 + 3x_3 + s_1 = 7$

$-2x_1 + 4x_2 + s_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ $s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

		$C_j \rightarrow$	-1	3	-2	0	0		
Basic Variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2		
s_1	0	7	3	-1	3	1	0	0	-
s_2	0	12	-2	[4]	0	0	1	0	3→
s_3	0	10	-4	3	8	0	0	1	10/3
				↑					
	Z' = 0		1	-3	2	0	0	0	←Δ_j
			$(R_1 = R_1 + R_2)$						
s_1	0	10	[5/2]	0	3	1	1/4	0	4→
			$(R_2 = R_2 / 4)$						
x_2	3	3	-1/2	1	0	0	1/4	0	-
			$(R_3 = R_3 - 3R_2)$						
s_3	0	1	-5/2	0	8	0	-3/4	1	-
			↑						
	Z' = 9		-5/2	0	0	0	3/4	0	←Δ_j
			$(R_1 = R_1 / 5/2)$						
x_1	-1	4	1	0	6/5	2/5	1/10	0	
			$(R_2 = R_2 + 1/2\ R_1)$						
x_2	3	5	0	1	3/5	1/5	3/10	0	
			$(R_3 = R_3 + 5/2R_1)$						
s_3	0	11	0	1	11	1	-1/2	1	

	Z' = 11	0	0	3/5	1/5	1/5	0	$\leftarrow\Delta_j$

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Z' =11 which implies Z = -11, $x_1 = 4$, $x_2 = 5$, $x_3 = 0$

Example 7
Max Z = 2x + 5y
$x + y \leq 600$
$0 \leq x \leq 400$
$0 \leq y \leq 300$

Solution

SLPP
Max $Z = 2x + 5y + 0s_1 + 0s_2 + 0s_3$
$x + y + s_1 = 600$
$x + s_2 = 400$
$y + s_3 = 300$
$x_1 \geq 0, y \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

		$C_j \rightarrow$	2	5	0	0	0	
Basic Variables	C_B	X_B	X	Y	S_1	S_2	S_3	Min ratio X_B / X_k
s_1	0	600	1	1	1	0	0	600 / 1 = 600
s_2	0	400	1	0	0	1	0	-
s_3	0	300	0	[1]	0	0	1	300 /1 = 300→
				↑				
	Z = 0		-2	-5	0	0	0	$\leftarrow\Delta_j$
				(R1 = R1 – R3)				
s_1	0	300	[1]	0	1	0	-1	300 /1 = 300→
s_2	0	400	1	0	0	1	0	400 / 1 = 400
y	5	300	0	1	0	0	1	-
			↑					
	Z = 1500		-2	0	0	0	5	$\leftarrow\Delta_j$
x	2	300	1	0	1	0	-1	
			(R2 = R2 – R1)					
s_2	0	100	0	0	-1	1	1	
y	5	300	0	1	0	0	1	

	Z = 2100	0	0	2	0	3	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Z = 2100, x = 300, y = 300

2.1 Computational Procedure of Big – M Method (Charne's Penalty Method)

Step 1 – Express the problem in the standard form.

Step 2 – Add non-negative artificial variable to the left side of each of the equations corresponding to the constraints of the type '≥' or '='.

When artificial variables are added, it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that artificial variables will be zero in the final solution (provided the solution of the problem exists).

On the other hand, if the problem does not have a solution, at least one of the artificial variables will appear in the final solution with positive value. This is achieved by assigning a very **large price (per unit penalty)** to these variables in the objective function. Such large price will be designated by –M for maximization problems (+M for minimizing problem), where M > 0.

Step 3 – In the last, use the artificial variables for the starting solution and proceed with the usual simplex routine until the optimal solution is obtained.

2.2 Worked Examples

Example 1

Max $Z = -2x_1 - x_2$

Subject to

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Max $Z = -2x_1 - x_2 + 0s_1 + 0s_2 - M a_1 - M a_2$

Subject to

$3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - s_1 + a_2 = 6$

$x_1 + 2x_2 + s_2 = 4$
$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$

Basic Variables	C_B	$C_j \rightarrow$ X_B	-2 X_1	-1 X_2	0 S_1	0 S_2	-M A_1	-M A_2	Min ratio X_B/X_k
a_1	-M	3	[3]	1	0	0	1	0	3 /3 = 1→
a_2	-M	6	4	3	-1	0	0	1	6 / 4 =1.5
s_2	0	4	1	2	0	1	0	0	4 / 1 = 4
	Z = -9M		↑ 2 – 7M	1 – 4M	M	0	0	0	←Δ_j
x_1	-2	1	1	1/3	0	0	X	0	1/1/3 =3
a_2	-M	2	0	[5/3]	-1	0	X	1	6/5/3 =1.2→
s_2	0	3	0	5/3	0	1	X	0	4/5/3=1.8
	Z = -2 – 2M			↑ $\frac{(-5M+1)}{3}$	0	0	X	0	←Δ_j
x_1	-2	3/5	1	0	1/5	0	X	X	
x_2	-1	6/5	0	1	-3/5	0	X	X	
s_2	0	1	0	0	1	1	X	X	
	Z = -12/5		0	0	1/5	0	X	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = -12/5, $x_1 = 3/5$, $x_2 = 6/5$

Example 2

Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 \geq 2$
$x_1 + 3x_2 \leq 3$
$x_2 \leq 4$
and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Max $Z = 3x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 - M a_1$

Subject to

$2x_1 + x_2 - s_1 + a_1 = 2$
$x_1 + 3x_2 + s_2 = 3$
$x_2 + s_3 = 4$
$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min ratio X_B/X_k
		$C_j \rightarrow$	3	-1	0	0	0	-M	
a_1	-M	2	[2]	1	-1	0	0	1	2 / 2 = 1→
s_2	0	3	1	3	0	1	0	0	3 / 1 = 3
s_3	0	4	0	1	0	0	1	0	-
	Z = -2M		↑ -2M-3	-M+1	M	0	0	0	←Δ_j
x_1	3	1	1	1/2	-1/2	0	0	X	-
s_2	0	2	0	5/2	1/2	1	0	X	2/1/2 = 4→
s_3	0	4	0	1	0	0	1	X	-
	Z = 3		0	5/2	↑ -3/2	0	0	X	←Δ_j
x_1	3	3	1	3	0	1/2	0	X	
s_1	0	4	0	5	1	2	0	X	
s_3	0	4	0	1	0	0	1	X	
	Z = 9		0	10	0	3/2	0	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 9, $x_1 = 3$, $x_2 = 0$

Example 3

Min $Z = 2x_1 + 3x_2$

Subject to

$x_1 + x_2 \geq 5$

$x_1 + 2x_2 \geq 6$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Min Z = Max Z2- = ′x_1 - $3x_2 + 0s_1 + 0s_2 - M a_1 - M a_2$

Subject to

$x_1 + x_2 – s_1 + a_1 = 5$

$x_1 + 2x_2 – s_2 + a_2 = 6$

$x_1 , x_2 , s_1, s_2, a_1, a_2 \geq 0$

		$C_j \rightarrow$	-2	-3	0	0	-M	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-M	5	1	1	-1	0	1	0	5 /1 = 5
a_2	-M	6	1	[2]	0	-1	0	1	6 / 2 = 3→
	Z11- = ′M		-2M + 2	-3M+3 ↑	M	M	0	0	←Δ_j
a_1	-M	2	[1/2]	0	-1	1/2	1	x	2/1/2 = 4→
x_2	-3	3	1/2	1	0	-1/2	0	x	3/1/2 =6
	Z2- = ′M-9		(-M+1) / 2 ↑	0	M	(-M+3)/2	0	x	←Δ_j
x_1	-2	4	1	0	-2	1	x	x	
x_2	-3	1	0	1	1	-1	x	x	
	Z11- = ′		0	0	1	1	x	x	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Z' = -11 which implies Max Z = 11, $x_1 = 4$, $x_2 = 1$

Example 4

Max $Z = 3x_1 + 2x_2 + x_3$

Subject to

$2x_1 + x_2 + x_3 = 12$

$3x_1 + 4x_2 = 11$

and x_1 is unrestricted

$x_2 \geq 0, x_3 \geq 0$

Solution

SLPP

Max $Z = 3(x_1' - x_1'') + 2x_2 + x_3 - M a_1 - M a_2$

Subject to

$2(x_1' - x_1'') + x_2 + x_3 + a_1 = 12$

$3(x_1' - x_1'') + 4x_2 + a_2 = 11$

$x_1', x_1'', x_2, x_3, a_1, a_2 \geq 0$

Max $Z = 3x_1' - 3x_1'' + 2x_2 + x_3 - M a_1 - M a_2$

Subject to

$2x_1' - 2x_1'' + x_2 + x_3 + a_1 = 12$

$3x_1' - 3x_1'' + 4x_2 + a_2 = 11$

$x_1', x_1'', x_2, x_3, a_1, a_2 \geq 0$

Basic Variables	C_B	X_B	X_1'	X_1''	X_2	X_3	A_1	A_2	Min ratio X_B/X_k
		$C_j \rightarrow$	3	-3	2	1	-M	-M	
a_1	-M	12	2	-2	1	1	1	0	12 /2 = 6
a_2	-M	11	3	-3	4	0	0	1	11/3 =3.6→
	Z = -23M		↑ -5M-3	5M+3	-5M-2	-M-1	0	0	←Δ_j
a_1	-M	14/3	0	0	-5/3	1	1	X	14/3/1 = 14/3→
x_1	3	11/3	1	-1	4/3	0	0	X	-
	$Z = \frac{-14M+11}{3}$		0	-6	5/3M+1	↑ -M-1	0	X	←Δ_j
x_3	1	14/3	0	0	-5/3	1	X	X	
x_1	3	11/3	1	-1	4/3	0	X	X	
	Z = 47/3		0	0	1/3	0	X	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

$x_1' = 11/3, \; x_1'' = 0$

$x_1 = x_1' - x_1'' = 11/3 - 0 = 11/3$

Therefore the solution is Max Z = 47/3, $x_1 = 11/3$, $x_2 = 0$, $x_3 = 14/3$

Example 5

Max $Z = 8x_2$

Subject to

$x_1 - x_2 \geq 0$

$2x_1 + 3x_2 \leq -6$

and x_1 , x_2 unrestricted

Solution

SLPP

Max $Z = 8(x_2' - x_2'') + 0s_1 + 0s_2 - M a_1 - M a_2$

Subject to

$(x_1' - x_1'') - (x_2' - x_2'') - s_1 + a_1 = 0$

$-2(x_1' - x_1'') - 3(x_2' - x_2'') - s_2 + a_2 = 6$

$x_1', x_1'', x_2', x_2'', s_1, a_1, a_2 \geq 0$

Max $Z = 8x_2' - 8x_2'' + 0s_1 + 0s_2 - M a_1 - M a_2$

Subject to

$x_1' - x_1'' - x_2' + x_2'' - s_1 + a_1 = 0$

$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' - s_2 + a_2 = 6$

$x_1', x_1'', x_2', x_2'', s_1, a_1, a_2 \geq 0$

	$C_j \rightarrow$		0	0	8	-8	0	0	-M	-M	
Basic Variables	C_B	X_B	X_1'	X_1''	X_2'	X_2''	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-M	0	1	-1	-1	1	-1	0	1	0	0→
a_2	-M	6	-2	2	-3	3	0	-1	0	1	2
	Z = -6M		M	-M	4M-8	-4M+8 ↑	M	M	0	0	←Δ_j
x_2''	-8	0	1	-1	-1	1	-1	0	X	0	-
a_2	-M	6	-5	[5]	0	0	3	-1	X	1	6/5→
	Z = -6M		5M-8	-5M+8 ↑	0	0	-3M+8	M	X	0	←Δ_j
x_2''	-8	6/5	0	0	-1	1	-2/5	-1/5	X	X	
x_1''	0	6/5	-1	1	0	0	3/5	-1/5	X	X	
	Z = -48/5		0	0	0	0	16/5	8/5	X	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

$x_1' = 0$, $x_1'' = 6/5$

$x_1 = x_1' - x_1'' = 0 - 6/5 = -6/5$

$x_2' = 0$, $x_2'' = 6/5$
$x_2 = x_2' - x_2'' = 0 - 6/5 = -6/5$

Therefore the solution is Max Z = -48/5, $x_1 = -6/5$, $x_2 = -6/5$

2.3 Steps for Two-Phase Method

The process of eliminating artificial variables is performed in **phase-I** of the solution and **phase- II** is used to get an optimal solution. Since the solution of LPP is computed in two phases, it is called as **Two-Phase Simplex Method**.

Phase I – In this phase, the simplex method is applied to a specially constructed **auxiliary linear programming problem** leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1 – Assign a cost -1 to each artificial variable and a cost 0 to all other variables in the objective function.

Step 2 – Construct the Auxiliary LPP in which the new objective function Z* is to be maximized subject to the given set of constraints.

Step 3 – Solve the auxiliary problem by simplex method until either of the following three possibilities do arise

i. Max Z* < 0 and atleast one artificial vector appear in the optimum basis at a positive level ($\Delta_j \geq 0$). In this case, given problem does not possess any feasible solution.
ii. Max Z* = 0 and at least one artificial vector appears in the optimum basis at a zero level. In this case proceed to phase-II.
iii. Max Z* = 0 and no one artificial vector appears in the optimum basis. In this case also proceed to phase-II.

Phase II – Now assign the actual cost to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints.

Simplex method is applied to the modified simplex table obtained at the end of phase-I, until an optimum basic feasible solution has been attained. The artificial variables which are non-basic at the end of phase-I are removed.

2.4 Worked Examples

Example 1

Max Z = $3x_1 - x_2$

Subject to

$2x_1 + x_2 \geq 2$
$x_1 + 3x_2 \leq 2$
$x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 - s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 2$

$x_2 + s_3 = 4$

$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 - 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1$

Subject to

$2x_1 + x_2 - s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 2$

$x_2 + s_3 = 4$

$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$

Phase I

		$C_j \rightarrow$	0	0	0	0	0	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min ratio X_B / X_k
a_1	-1	2	[2]	1	-1	0	0	1	1→
s_2	0	2	1	3	0	1	0	0	2
s_3	0	4	0	1	0	0	1	0	-
	$Z^* = -2$		↑ -2	-1	1	0	0	0	←Δ_j
x_1	0	1	1	1/2	-1/2	0	0	x	
s_2	0	1	0	5/2	1/2	1	0	x	
s_3	0	4	0	1	0	0	1	x	
	$Z^* = 0$		0	0	0	0	0	x	←Δ_j

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

		$C_j \rightarrow$	3	-1	0	0	0

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k
x_1	3	1	1	1/2	-1/2	0	0	-
s_2	0	1	0	5/2	1/2	1	0	2→
s_3	0	4	0	1	0	0	1	-
	Z = 3		0	5/2	↑ -3/2	0	0	←Δ_j
x_1	3	2	1	3	0	1	0	
s_1	0	2	0	5	1	2	0	
s_3	0	4	0	1	0	0	1	
	Z = 6		0	10	0	3	0	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 6, $x_1 = 2$, $x_2 = 0$

Example 2

Max $Z = 5x_1 + 8x_2$

Subject to

$3x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$

$x_1 + x_2 \leq 5$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

Standard LPP

Max $Z = 5x_1 + 8x_2$

Subject to

$3x_1 + 2x_2 - s_1 + a_1 = 3$

$x_1 + 4x_2 - s_2 + a_2 = 4$

$x_1 + x_2 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1 - 1a_2$

Subject to

$3x_1 + 2x_2 - s_1 + a_1 = 3$

$x_1 + 4x_2 - s_2 + a_2 = 4$

$x_1 + x_2 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0$

Phase I

C_j →	0	0	0	0	0	-1	-1

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	A_2	Min ratio X_B / X_k
a_1	-1	3	3	2	-1	0	0	1	0	3/2
a_2	-1	4	1	[4]	0	-1	0	0	1	1→
s_3	0	5	1	1	0	0	1	0	0	5
	$Z^* = -7$		-4	↑ -6	1	1	0	0	0	←Δ_j
a_1	-1	1	[5/2]	0	-1	1/2	0	1	x	2/5→
x_2	0	1	1/4	1	0	-1/4	0	0	x	4
s_3	0	4	3/4	0	0	1/4	1	0	x	16/3
	$Z^* = -1$		↑ -5/2	0	1	-1/2	0	0	x	←Δ_j
x_1	0	2/5	1	0	-2/5	1/5	0	x	x	
x_2	0	9/10	0	1	1/10	-3/10	0	x	x	
s_3	0	37/10	0	0	3/10	1/10	1	x	x	
	$Z^* = 0$		0	0	0	0	0	x	x	←Δ_j

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

$C_j \rightarrow$ 5 8 0 0 0

Basic Min ratio

Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	X_B/X_k
x_1	5	2/5	1	0	-2/5	1/5	0	2→
x_2	8	9/10	0	1	1/10	-3/10	0	-
s_3	0	37/10	0	0	3/10	1/10	1	37
						↑		
	Z = 46/5		0	0	-6/5	-7/5	0	←Δ_j
s_2	0	2	5	0	-2	1	0	-
x_2	8	3/2	3/2	1	-1/2	0	0	-
s_3	0	7/2	-1/2	0	1/2	0	1	7→
					↑			
	Z = 12		7	0	-4	0	0	←Δ_j
s_2	0	16	3	0	0	1	2	
x_2	8	5	1	1	0	0	1/2	
s_1	0	7	-1	0	1	0	2	
	Z = 40		3	0	0	0	4	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 40, $x_1 = 0$, $x_2 = 5$

Example 3

Max $Z = -4x_1 - 3x_2 - 9x_3$

Subject to

$2x_1 + 4x_2 + 6x_3 \geq 15$

$6x_1 + x_2 + 6x_3 \geq 12$

and $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$

Solution

Standard LPP

Max $Z = -4x_1 - 3x_2 - 9x_3$

Subject to

$2x_1 + 4x_2 + 6x_3 - s_1 + a_1 = 15$

$6x_1 + x_2 + 6x_3 - s_2 + a_2 = 12$

x_1 , x_2 , s_1, s_2, a_1, $a_2 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 - 0x_2 - 0x_3 + 0s_1 + 0s_2 - 1a_1 - 1a_2$

Subject to

$2x_1 + 4x_2 + 6x_3 - s_1 + a_1 = 15$

$6x_1 + x_2 + 6x_3 - s_2 + a_2 = 12$

x_1 , x_2 , s_1, s_2, a_1, $a_2 \geq 0$

Phase I

	$C_j \rightarrow$	0	0	0	0	0	-1	-1	
Basic									Min ratio

Variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	A_1	A_2	X_B/X_k
a_1	-1	15	2	4	6	-1	0	1	0	15/6
a_2	-1	12	6	1	6	0	-1	0	1	2→
					↑					
	Z* = -27		-8	-5	-12	1	1	0	0	←Δ_j
a_1	-1	3	-4	3	0	-1	1	1	x	1→
x_3	0	2	1	1/6	1	0	-1/6	0	x	12
				↑						
	Z* = -3		4	-3	0	1	-1	0	x	←Δ_j
x_2	0	1	-4/3	1	0	-1/3	1/3	x	x	
x_3	0	11/6	22/18	0	1	1/18	-4/18	x	x	
	Z* = 0		0	0	0	0	0	x	x	

Since all $\Delta_j \geq 0$, Max Z* = 0 and no artificial vector appears in the basis, we proceed to phase II.

Phase II

Basic Variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	Min ratio X_B/X_k
	$C_j \rightarrow$		-4	-3	-9	0	0	
x_2	-3	1	-4/3	1	0	-1/3	1/3	-
x_3	-9	11/6	22/18	0	1	1/18	-4/18	3/2→
			↑					
	Z = -39/2		-3	0	0	1/2	1	←Δ_j
x_2	-3	3	0	1	12/11	-3/11	1/11	
x_1	-4	3/2	1	0	18/22	1/22	-4/22	
	Z = -15		0	0	27/11	7/11	5/11	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = -15, $x_1 = 3/2$, $x_2 = 3$, $x_3 = 0$

Example 4

Min $Z = 4x_1 + x_2$

Subject to

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

and $x_1 \geq 0,\ x_2 \geq 0$

Solution

Standard LPP

Min Z = Max $Z' = -4x_1 - x_2$

Subject to

$3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - s_1 + a_2 = 6$

$x_1 + 2x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 - 0x_2 + 0s_1 + 0s_2 - 1a_1 - 1a_2$

Subject to

$3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - s_1 + a_2 = 6$

$x_1 + 2x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2, a_1, a_2 \geq 0$

Phase I

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min ratio X_B/X_k
	$C_j \rightarrow$		0	0	0	0	-1	-1	
a_1	-1	3	[3]	1	0	0	1	0	1→
a_2	-1	6	4	3	-1	0	0	1	6/4
s_2	0	4	1	2	0	1	0	0	4
	$Z^* = -9$		↑ -7	-4	1	0	0	0	
x_1	0	1	1	1/3	0	0	X	0	3
a_2	-1	2	0	[5/3]	-1	0	X	1	6/5→
s_2	0	3	0	5/3	0	1	X	0	9/5
	$Z^* = -2$		0	↑ -5/3	1	0	X	0	
x_1	0	3/5	1	0	1/5	0	X	X	
x_2	0	6/5	0	1	-3/5	0	X	X	
s_2	0	1	0	0	1	1	X	X	
	$Z^* = 0$		0	0	0	0	X	X	

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B/X_k
	$C_j \rightarrow$		-4	-1	0	0	
x_1	-4	3/5	1	0	1/5	0	3
x_2	-1	6/5	0	1	-3/5	0	-
s_2	0	1	0	0	[1]	1	1→
	$Z' = -18/5$		0	0	↑ -1/5	0	←Δ_j
x_1	-4	2/5	1	0	0	-1/5	
x_2	-1	9/5	0	1	0	3/5	
s_1	0	1	0	0	1	1	
	$Z' = -17/5$		0	0	0	1/5	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z' = -17/5$
Min $Z = 17/5$, $x_1 = 2/5$, $x_2 = 9/5$

Example 5
Min $Z = x_1 - 2x_2 - 3x_3$
Subject to
$-2x_1 + x_2 + 3x_3 = 2$
$2x_1 + 3x_2 + 4x_3 = 1$
and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP
Min $Z = $ Max $Z' = -x_1 + 2x_2 + 3x_3$
Subject to
$-2x_1 + x_2 + 3x_3 + a_1 = 2$
$2x_1 + 3x_2 + 4x_3 + a_2 = 1$
$x_1, x_2, a_1, a_2 \geq 0$

Auxiliary LPP
Max $Z^* = 0x_1 + 0x_2 + 0x_3 - 1a_1 - 1a_2$
Subject to
$-2x_1 + x_2 + 3x_3 + a_1 = 2$
$2x_1 + 3x_2 + 4x_3 + a_2 = 1$
$x_1, x_2, a_1, a_2 \geq 0$

Phase I

		$C_j \rightarrow$	0	0	0	-1	-1	
Basic Variables	C_B	X_B	X_1	X_2	X_3	A_1	A_2	Min Ratio X_B / X_K
a_1	-1	2	-2	1	3	1	0	2/3
a_2	-1	1	2	3	[4]	0	1	1/4→
					↑			
	$Z^* = -3$		0	-4	-7	0	0	$\leftarrow \Delta_j$
a_1	-1	5/4	-7/4	-5/4	0	1	x	
x_3	0	1/4	1/2	3/4	1	0	x	
	$Z^* = -5/4$		7/4	5/4	0	1	x	$\leftarrow \Delta_j$

Since for all $\Delta_j \geq 0$, optimum level is achieved. At the end of phase-I Max Z* < 0 and one of the artificial variable a_1 appears at the positive optimum level. Hence the given problem does not posses any feasible solution.

3.1.1 Degeneracy

The concept of obtaining a degenerate basic feasible solution in a LPP is known as degeneracy. The degeneracy in a LPP may arise

- At the initial stage when at least one basic variable is zero in the initial basic feasible solution.
- At any subsequent iteration when more than one basic variable is eligible to leave the basic and hence one or more variables becoming zero in the next iteration and the problem is said to degenerate. There is no assurance that the value of the objective function will improve, since the new solutions may remain degenerate. As a result, it is possible to repeat the same sequence of simplex iterations endlessly without improving the solutions. This concept is known as cycling or circling.

Rules to avoid cycling

- Divide each element in the tied rows by the positive coefficients of the key column in that row.
- Compare the resulting ratios, column by column, first in the identity and then in the body, from left to right.
- The row which first contains the smallest algebraic ratio contains the leaving variable.

Example 1

Max $Z = 3x_1 + 9x_2$
Subject to
$x_1 + 4x_2 \leq 8$
$x_1 + 2x_2 \leq 4$
and $x_1 \geq 0,\ x_2 \geq 0$

Solution

Standard LPP
Max $Z = 3x_1 + 9x_2 + 0s_1 + 0s_2$
Subject to
$x_1 + 4x_2 + s_1 = 8$
$x_1 + 2x_2 + s_2 = 4$
$x_1,\ x_2,\ s_1,\ s_2 \geq 0$

		$C_j \rightarrow$	3	9	0	0		
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	X_B / X_K	S_1 / X_2
							2 }	
				↑			2 }	

s_1	0	8	1	4	1	0		1/4
s_2	0	4	1	2	0	1		0/2→
				↑				
	Z = 0		-3	-9	0	0		←Δ_j
s_1	0	0	-1	0	1	-1		
x_2	9	2	1/2	1	0	1/2		
	Z =18		3/2	0	0	9/2		

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 18, $x_1 = 0$, $x_2 = 2$

Note – Since a tie in minimum ratio (degeneracy), we find minimum of s_1 / x_k for these rows for which the tie exists.

Example 2

Max $Z = 2x_1 + x_2$
Subject to
$4x_1 + 3x_2 \leq 12$
$4x_1 + x_2 \leq 8$
$4x_1 - x_2 \leq 8$
and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP
Max $Z = 2x_1 + x_2 + 0s_1 + 0s_2 + 0s_3$
Subject to
$4x_1 + 3x_2 + s_1 = 12$
$4x_1 + x_2 + s_2 = 8$
$4x_1 - x_2 + s_3 = 8$
$x_1, x_2, s_1, s_2, s_3 \geq 0$

		$C_j \rightarrow$	2	1	0	0	0			
Basic Varibles	C_B	X_B	X_1	X_2	S_1	S_2	S_3	X_B / X_K	S_1 / X_1	S_2 / X_1
s_1	0	12	4	3	1	0	0	12/4=3		
s_2	0	8	4	1	0	1	0	8/4=2	4/0=0	1/4
s_3	0	8	[4]	-1	0	0	1	8/4=2	4/0=0	0/4=0→
			↑							
	Z = 0		-2	-1	0	0	0	←Δ_j		
1	0	4	0	4	1	0	-1	4/4=1		
s_2	0	0	0	2	0	1	-1	0→		
x_1	2	2	1	-1/4	0	0	1/4	-		
				↑						
	Z = 4		0	-3/2	0	0	1/2	←Δ_j		
s_1	0	4	0	0	1	-2	[1]	0→		
x_2	1	0	0	1	0	1/2	-1/2	-		
x_1	2	2	1	0	0	1/8	1/8	16		
							↑			
	Z = 4		0	0	0	3/4	-1/4	←Δ_j		
s_3	0	4	0	0	1	-2	1			
x_2	1	2	0	1	1/2	-1/2	0			
x_1	2	3/2	1	0	-1/8	3/8	0			
	Z = 5		0	0	1/4	1/4	0	←Δ_j		

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 5, $x_1 = 3/2$, $x_2 = 2$

3.1.2 Non-existing Feasible Solution

The feasible region is found to be empty which indicates that the problem has no feasible solution.

Example

Max $Z = 3x_1 + 2x_2$

Subject to

$2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$ and $x_1 \geq 0$, $x_2 \geq 0$

Solution

Standard LPP

Max $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - Ma_1$

Subject to

$2x_1 + x_2 + s_1 = 2$

$3x_1 + 4x_2 - s_2 + a_1 = 12$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

		$C_j \rightarrow$	3	2	0	0	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	Min Ratio X_B / X_K
s_1	0	2	2	[1]	1	0	0	2/1=2→
a_1	-M	12	3	4	0	-1	1	12/4=3
				↑				
	Z= -12M		-3M-3	-4M-2	0	M	0	←Δ_j
x_2	2	2	2	1	1	0	0	
a_1	-M	4	-5	0	-4	-1	1	
	Z= 4-4M		1+5M	0	2+4M	M	0	

$\Delta_j \geq 0$ so according to optimality condition the solution is optimal but the solution is called **pseudo optimal solution** since it does not satisfy all the constraints but satisfies the optimality condition. The artificial variable has a positive value which indicates there is no feasible solution.