

MODULE – IV

PROBABILITY AND JOINT PROBABILITY

Probability = $\frac{\text{The total number of favourable outcomes}}{\text{The total number of outcomes}}$

Example

If we toss a coin we get either head or tail

$$S = \{H, T\}$$

Probability of getting head = $1/2$

Probability of getting tail = $1/2$

Example

If we throw a die we get $S = \{1,2,3,4,5,6\}$

Probability of getting one = $1/6$

Probability of getting an odd number = $3/6$

Probability of getting an even number = $3/6$

Random Variable

A variable whose value is determined by the outcome of a random experiment is called a random variable. A random variable is also known as a stochastic variable.

Random Variable example

Consider tossing of two coins

We get $S = \{HH,HT,TH,TT\}$

Heads (X)	0	1	2	Total out comes
Number of occurrence	1	2	1	4
P(x)	$1/4$	$2/4$	$1/4$	

Random Variable example

Consider tossing of three coins

We get $S = \{HHH,HTT,THH,TTH,HHT,HTH,THT,TTT\}$

Heads (X)	0	1	2	3	Total out comes
Number of occurrence	1	3	3	1	8
P(x)	1/8	3/8	3/8	1/8	

There are two kinds of random variables

1. Discrete random variables
2. Continuous random variable

If the random variable takes on the integer values such as 0,1,2,3----- then it is called discrete random variable.

Example : The number of telephone calls received by a person.

If the random variable takes on all values in a certain interval then it is called continuous random variable.

Example : Height of the students in a class

Probability distribution function

If X is a discrete random variable which takes the values x_1, x_2, x_3, \dots with probabilities p_1, p_2, p_3, \dots . Then p_i is called the probability function which satisfies the following conditions

1. $p_i \geq 0$ for all i
2. $\sum_{i=1} p_i = 1$

For discrete random variable

$$\text{Mean} = \mu = E(X) = \sum x p(x)$$

$$\text{Variance} = \sigma^2 = \sum x^2 p(x) - \mu^2$$

$$\text{Standard deviation} = \sqrt{\text{Variance}} = \sigma$$

1. The probability density function of a variate X is Find k, $p(X \geq 5)$, $p(3 < X \leq 6)$, Mean, Variance

X	0	1	2	3	4	5	6
P(X)	k	2k	5k	7k	9k	11k	13k

(Answer: 1/49, 24/49, 33/49, 202/49, 6824/2401)

2. The probability density function of a variate X is Find k, $p(X < 6)$, $p(X \geq 6)$, $p(3 < X \leq 6)$

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

(Answer: 1/10, 81/100, 19/100, 33/100)

3. The probability density function of a variate X is Find k, Mean, Variance, SD

X	-2	-1	0	1	2	3
P(X)	0.1	k	0.2	2k	0.3	k

(Answer: 0.1, 0.8, 2.16, 1.47)

Important points to be remember

Exactly : Exactly 3 means $P(X = 3)$

At least : At least 3 means $P(X \geq 3)$

$$= 1 - P(X < 3)$$

At most : At most 3 means $P(X \leq 3)$

Binomial Distribution

The probability distribution function is $P(x) = {}^n C_x p^x q^{n-x}$ with $p + q = 1$

$$(p + q)^n \text{ i.e. } \sum_{x=0}^n {}^n C_x p^x q^{n-x} = 1$$

1. Let x be a binomial distributed random variable with mean 2 and standard deviation $2/\sqrt{3}$. Find the corresponding probability density function.

$$\text{Answer : } P(x) = {}^6 C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

2. Let x be a binomial distributed random variable based on 6 Bernoullian trails. If $p = 0.3$ evaluate $P(x = 3)$, $P(x \leq 3)$, $P(x = 4)$ and $P(x > 4)$

$$\text{Answer : } P(x) = {}^6 C_x (0.3)^x (0.7)^{6-x}$$

$$0.1852, 0.9295, 0.0595, 0.011$$

3. Six fair coins are tossed. Find the probability of getting (i) exactly 3 heads (ii) at least 3 heads (iii) at least one head

$$\text{Answer : } P(x) = {}^6 C_x (0.5)^x (0.5)^{6-x}$$

$$0.3125, 0.6563, 0.9844$$

4. In a consignment of electric lamps 5% are defective. If a random sample of 8 lamps are inspected what is the probability that one are more lamps are defective.

$$\text{Answer : } P(x) = {}^8 C_x (0.05)^x (0.95)^{8-x}$$

$$0.3366$$

5. The probability that a person aged 60 years will live up to 70 is 0.65. what is the probability that out of 10 person aged 60 at least 7 of them will live up to 70

$$\text{Answer : } P(x) = {}^{10} C_x (0.65)^x (0.35)^{10-x}$$

$$0.5139$$

6. The number of telephone lines at an instant of time is a binomial variate with probability 0.2 that a line is busy. If 10 lines are chosen at random, what is the probability that (i) no line is busy (ii) 5 lines are busy (iii) at least one line is busy (iv) at most 2 lines are busy (v) all lines are busy

Answer : $P(x) = {}^{10}C_x (0.2)^x (0.8)^{10-x}$

(i) 0.1074, (ii) 0.02642, (iii) 0.8926, (iv) 0.6778, (v) 1.024×10^{-7}

7. In a large number of parts manufactured by a machine, the mean number of defective in a sample of 20 is 2. Out of 1000 such samples how many would be expected to contain at least 3 defective parts.

Answer : $P(x) = {}^{20}C_x (0.1)^x (0.9)^{20-x}$

0.323 $1000 \times 0.323 = 323$

X	0	1	2	3	4
f	5	29	36	25	5

8. In 256 sets of 12 tosses of an honest coin, in how many sets one can expect 8 heads and 4 tails.

Answer : $P(x) = {}^{12}C_x (0.5)^x (0.5)^{12-x}$ 0.1208 $256 \times 0.1208 = 31$

9. Fit a binomial distribution for the frequency distribution. Also calculate the theoretical frequencies

Answer : $P(x) =$

Theoretical frequencies = ${}^4C_x (0.49)^x (0.51)^{4-x}$

$$= \frac{100({}^4C_x (0.49)^x (0.51)^{4-x})}{(\sum f)p(x)}$$

$$\sum f = 5 + 29 + 36 + 25 + 5 = 100$$

$$\sum fx = 0 + 29 + 72 + 75 + 20 = 196$$

$$\text{Mean} = np = \frac{\sum fx}{\sum f} = 1.96$$

$$p = \frac{1.96}{4} = 0.49$$

$$x=0 \quad {}^4C_0 (0.49)^0 (0.51)^4 = 6.76$$

$$x=1 \quad {}^4C_1 (0.49)^1 (0.51)^3 = 26$$

$$x=2 \quad {}^4C_2 (0.49)^2 (0.51)^2 = 37.47$$

$$x=3 \quad {}^4C_3 (0.49)^3 (0.51)^1 = 24$$

$$x=4 \quad {}^4C_4 (0.49)^4 (0.51)^0 = 5.76$$

10. Fit a binomial distribution for the frequency distribution. Also calculate the theoretical frequencies

X	0	1	2	3	4	5
f	2	14	20	34	22	8

Answer : $P(x) = {}^5C_x (0.57)^x (0.43)^{5-x}$ 1.5, 10.45, 26, 34.2, 22.48, 5.91

Poisson's Distribution

The probability distribution function is

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad x > 0$$

1. Find the Poisson's probability distribution which has mean 2. Also find $P(X = 4)$.

Ans: $p(x) = \frac{e^{-2} 2^x}{x!}$ $P(x=4) = 0.0902$

2. If 1% of switches manufactured by a firm are found to be defective. Find the probability that a box contains 200 switches (i) no defective (ii) 3 or more defective switches.

(ANS.0.0183, 0.7621)

3. Ten percent of the tools produced in a certain manufacturing process turn out to be defective. Find the probability that in a sample of 10 tools chosen at random, exactly 2 will be defective.

(ANS.0.18)

4. If the probability that an individual will suffer a bad reaction from injection of a given serum is 0.001, determine that out of 2000 individuals (i) exactly 3 (ii) more than 2 individuals will suffer a bad reaction.

(ANS. 0.18, 0.323)

5. In a certain factory manufacturing razor blades there is a small chance 1/50 for any blade to be defective the blades are placed in pockets each containing 10 blades. Calculate the approximate number of pockets containing (i) no defective (ii) one defective (iii) two defective blades in a consignment of 10000 pockets. Also calculate the approximate number of pockets containing not more than 2 defective blades in a consignment of 10000 pockets.

(ANS. 8187,1637,163, 12)

6. The number of accidents in a year to taxi drivers in a city follows a poisson's distribution with mean 3. Out of 1000 taxi drivers find approximately the number of drivers with (i) no accidents (ii) more than 3 accidents in a year.

(ANS. 50,350)

7. Fit a Poisson distribution for the frequency distribution. Also calculate the theoretical frequencies

X	0	1	2	3	4
f	122	60	15	2	1

$$\sum f = 122 + 60 + 15 + 2 + 1 = 200$$

$$\sum fx = 0 + 60 + 30 + 6 + 4 = 100$$

$$\text{Mean} = np = \frac{\sum fx}{\sum f} = 0.5$$

$$\lambda = 0.5$$

$$\text{Answer : } P(x) = \frac{e^{-0.5} 0.5^x}{x!}$$

$$\text{Theoretical frequencies} = (\sum f) p(x) = 200 \left(\frac{e^{-0.5} 0.5^x}{x!} \right)$$

$$x=0 \quad 200 \frac{e^{-0.5} 0.5^0}{0!} = 121.3$$

$$x=1 \quad 200 \frac{e^{-0.5} 0.5^1}{1!} = 60.65$$

$$x=2 \quad 200 \frac{e^{-0.5} 0.5^2}{2!} = 15.1625$$

$$x=3 \quad 200 \frac{e^{-0.5} 0.5^3}{3!} = 2.527$$

$$x=4 \quad 200 \frac{e^{-0.5} 0.5^4}{4!} = 0.3159$$

Continuous Random Variable

$$P(x) \geq 0$$

$$P(-\infty < x < \infty) = \int_{-\infty}^{\infty} f(x) dx = 1 \quad P(a \leq x \leq b) = \int_a^b f(x) dx$$

$$\text{Mean} = \int_{-\infty}^{\infty} x f(x) dx$$

$$\text{Variance} = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

1. A random variable x has the density function $p(x) = \begin{cases} kx^2 & 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$

Find k, $P(x \leq 1)$, $P(1 \leq x \leq 3)$, $P(x \leq 2)$, $P(x > 1)$ and $P(x > 2)$

(Answers 1/9, 1/27, 7/27, 8/27, 26/27, 19/27)

2. A random variable x has the density function $p(x) = \begin{cases} kx^2 & 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$

Find k , Mean, Variance, Standard deviation

(Answers $9/4$, $27/80$, 0.5809)

3. A random variable x has the density function $p(x) = \begin{cases} kx & 0 \leq x \leq 5 \\ 0 & \text{elsewhere} \end{cases}$

Find k , $P(x \leq 3)$, $P(1 \leq x \leq 3)$, Mean, Variance, Standard deviation

(Answers $2/25$, $8/25$, $9/25$, $4/3$, $7/9$, 0.554)

4. A random variable x has the density function $p(x) = \frac{x}{1+x^2} \quad -\infty < x < \infty$

Find k , $P(x \geq 0)$, $P(0 \leq x \leq 1)$,

(Answers $1/\pi$, $1/2$, $1/4$)

Exponential Distribution

The PDF of Exponential distribution is $p(x) = \begin{cases} \alpha e^{-\alpha x} & 0 \leq x < \infty \\ 0 & \text{elsewhere} \end{cases}$

1. If x is an exponential variate with mean 5 evaluate the following $P(0 < x < 1)$, $P(x \leq 0 \text{ or } x \geq 1)$, $P(-\infty < x < 10)$

Answers 0.1812 , 0.8646 , 0.8187

2. If x is an exponential variate with mean 4 evaluate the following $P(0 < x < 1)$, $P(x > 2)$, $P(-\infty < x < 10)$

Answers 0.2212 , 0.6065 , 0.9179

3. In a certain town the duration of a shower is exponentially distributed with mean equal to 5 minutes, What is the probability that a shower will last for (i) less than 10 minutes (ii) 10 minutes or more

Answers 0.8647 , 0.1353 ,

4. The length of a telephone conversation has been exponentially distribution with mean of 3 minutes. Find the probability that a call (i) ends in less than 3 minutes and (ii) takes between 3 and 5 minutes

Answers 0.6321 , 0.179

5. At a certain city bus stop three buses arrive per hour on an average. Assuming that the time between successive arrivals is exponentially distributed, find the probability that the time between the arrival of successive buses is (i) less than 10 minutes and (ii) at least 30 minutes

Answers 0.3935 , 0.2231 ,

6. The daily turnover in a medical shop is exponentially distributed with Rs.6000 as the average with a net profit of 8%. Find the probability that the net profit exceeds Rs.500 on a randomly chosen day.

Let x denote the random variable denoting the turnover per day.

$$\text{Given } \frac{1}{\alpha} = 6000 \Rightarrow \alpha = \frac{1}{6000}$$

Let A be the turnover in rupees for which the net profit is Rs.500. Then since net profit is 8% of the turnover we have

$$\frac{8}{100} \times A = 500 \Rightarrow A = 6250$$

Since the profit exceed Rs.500 the turnover has to exceed Rs.6250. Hence the probability that the net profit exceeds Rs.500 is given by

$$\begin{aligned} P(x > 6250) &= 1 - P(x \leq 6250) \\ &= 1 - \int_0^{6250} p(x) dx \\ &= 1 - \int_0^{6250} \frac{1}{6000} e^{-1/6000x} dx \\ &= 0.353 \end{aligned}$$

7. The length of a telephone conversation has been exponentially distribution with mean of 3 minutes. Find the probability that a call (i) ends in more than 1 minutes and (ii) takes less than 3 minutes

Answers 0.71655, 0.63212

8. The daily turnover in a departmental store is exponentially distributed with Rs.10000 as the average with a net profit of 8%. Find the probability that the net profit exceeds Rs.300 on a randomly chosen two consecutive days.

Answer 0.00553

Normal Distribution

The PDF of Normal distribution is $P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$

Normal Distribution Key Points

$$z = \frac{x - \mu}{\sigma}$$

$$P(-\infty < x < \infty) = P(-\infty < z < \infty)$$

$$P(z \leq 0) = A(-\infty, 0) = A(0, \infty) = 0.5$$

$A(z)$ is the area of the curve

$$1. P(0 \leq z \leq 1.45) = A(1.45) = 0.4265$$

$$2. P(-2.60 \leq z \leq 0) = P(0 \leq z \leq 2.60) = A(2.60) = 0.4953$$

$$3. P(-3.40 \leq z \leq 2.65)$$

$$= P(-3.40 \leq z \leq 0) + P(0 \leq z \leq 2.65)$$

$$= P(0 \leq z \leq 3.40) + P(0 \leq z \leq 2.65)$$

$$= A(3.40) + A(2.65) = 0.49966 + 0.4960$$

$$= 0.99566$$

$$4. P(1.25 \leq z \leq 2.1) = A(2.1) - A(1.25) = 0.0877$$

$$5. P(-2.55 \leq z \leq -0.8) = P(0.8 \leq z \leq 2.55) = A(2.55) - A(0.8) = 0.2065$$

$$6. P(z \geq 1.7) = 0.5 - P(0 \leq z \leq 1.7) = 0.5 - A(1.7) = 0.0446$$

$$7. P(z \leq -3.35) = P(z \geq 3.35)$$

$$= 0.5 - P(0 \leq z \leq 3.35)$$

$$= 0.5 - A(3.35)$$

$$= 0.0004$$

$$8. P(|z| \leq 1.85) = P(-1.85 \leq z \leq 1.85)$$

$$= 2 A(1.85)$$

$$= 2(0.4678)$$

$$= 0.9356$$

9. For the normal distribution with mean 2 and standard deviation 4 evaluate the following probabilities

$$(i) P(-6 < x < 3) \quad (ii) P(1 < x < 5) \quad (iii) P(x \geq 5)$$

$$(iv) P(|x| < 4) \quad (v) P(|x| > 3) \quad (vi) P(|x - 2| > 1)$$

$$(0.5759, 0.3721, 0.2266, 0.6247, 0.5069, 0.4013)$$

10. The weekly wages of workers in a company are normally distributed with mean of Rs.700 and standard deviation Rs.50. Find the probability that the weekly wage of a randomly chosen worker is

$$(i) \text{ between Rs.650 and Rs. 750}$$

$$(ii) \text{ more than Rs.750}$$

Answer : 0.6826, 0.1587

11. In a certain city the number of power breakdowns per week is normal variate with mean 11.6 and standard deviation 3.3. Find the probability that there will be at least eight breakdowns in any week.

Answer: 0.8925

12. The I.Q. of students in a certain college is assumed to be normally distributed with mean 100 and variance 25. If two students are selected randomly find the probability that have I Q between 102 and 110.

Answer 0.54

13. The mean weight of 500 students at a certain school is 50 kgs and the standard deviation is 6 kg. Assuming that the weights are normally distributed, find the expected number of students weighing (i) between 40 and 50 kgs. And (ii) more than 60 kgs given that $A(1.6667) = 0.4525$

Answer 0.4525, 0.0475, 226, 24

14. The life of a certain type of electrical lamp is normally distributed with mean of 2040 hours and standard deviation 60 hours. In a consignment of 2000 lamps find how many would be expected to burn for (i) more than 1950 hours (ii) less than 1950 hours (iii) between 1920 hours and 2160 hours, given that $A(1.5) = 0.4332$ $A(1.83) = 0.4664$.

Answers: 0.0336, 0.0668, 0.9544, 67, 134, 1909

15. In an examination taken by 500 candidates the average and SD of marks obtained are 40% and 10% respectively. Assuming normal distribution find (i) how many have scored above 60% (ii) how many will pass if 50% is fixed as the minimum marks for passing (iii) how many will pass if 40% is fixed as the minimum marks for passing (iv) what should be the percentage of marks for passing so that 350 candidates pass. Answers: 0.0228, 0.1587, 0.5, 11, 79, 250, 35

16. In a normal distribution 31% of the items are under 45 and 8% are over 64. Find the mean and standard deviation, given that $A(0.5) = 0.19$ and $A(1.4) = 0.42$ where $A(z)$ is the area under the standard normal curve from 0 to $z > 0$.

Answers : Mean = $\mu = 50$ and SD = $\sigma = 10$

17. In a normal distribution 7% are under 35 and 89% are under 63. Find the mean and standard deviation, given that $A(1.23) = 0.39$ $A(1.48) = 0.43$ in the usual notations.

Answers : Mean = $\mu = 50.29$ and SD = $\sigma = 10.33$

18. An analog signal received as a detector (measured in microvolt's) may be modeled as a normal random variable with mean 200 and variance 256 at a fixed point of time. What is the probability that the signal will exceed 240 microvolt's.

Answer: 0.0062

19. A certain machine makes electric resistors having a mean of 40 ohms and standard deviation of 2 ohms. Assuming that the resistance follows a normal distribution. What is the percentage of resistors will have the resistance that exceeds 43 ohms.

Answer: 0.0668, 6.68%

20. Fit a normal distribution for the following frequency distribution

x	2	4	6	8	10
f	1	4	6	4	1

$$\sum f = 1 + 4 + 6 + 4 + 1 = 16$$

$$\sum x f = 96 \quad \sum x^2 f = 640$$

$$\bar{X} = \frac{\sum x f}{\sum f} = \frac{96}{16} = 6$$

$$\sigma^2 = \frac{\sum x^2 f}{\sum f} - \bar{X}^2 = 4$$

$$P(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-6)^2}{8}} \quad -\infty < x < \infty$$

Joint Probability Distribution

$$\text{Mean of } X = \mu_x = E(X) = \sum x p(x)$$

$$\text{Mean of } Y = \mu_y = E(Y) = \sum y p(y)$$

$$E(XY) = \sum_x \sum_y x y p(x, y)$$

X and Y are independent if

$$E(XY) = E(X)E(Y)$$

$$\text{Variance of } X = \sigma_x^2 = E(X^2) - [E(X)]^2$$

$$\text{Variance of } Y = \sigma_y^2 = E(Y^2) - [E(Y)]^2$$

$$\text{Standard Deviation of } X = \sigma_x = \sqrt{\text{Variance}}$$

$$\text{Standard Deviation of } Y = \sigma_y = \sqrt{\text{Variance}}$$

$$\begin{aligned} \text{Covariance of } X \text{ and } Y &= \text{Cov}(X, Y) \\ &= E(XY) - E(X)E(Y) \end{aligned}$$

$$\begin{aligned} \text{Correlation Coefficient} &= \rho(X, Y) \\ &= \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} \end{aligned}$$

X and Y are independent if $\text{Cov}(X, Y) = 0$

1. The joint probability distribution of two random variables X and Y is given by the following table. Determine the individual or marginal distributions of X and Y . Also verify X and Y are stochastically independent.

X/Y	2	3	4
1	0.06	0.15	0.09
2	0.14	0.35	0.21

Answers Means = 1.7 and 3.1 $E(XY) = 5.27$

2. The joint probability distribution of two random variables X and Y is given by the following table. Determine the individual or marginal distributions of X and Y . Also verify X and Y are stochastically independent.

X/Y	1	3	6
1	1/9	1/6	1/18
3	1/6	1/4	1/12
6	1/18	1/12	1/36

Answers $E(X) = 2.833 = E(Y)$ $E(XY) = 8.026$ $\text{Cov}(X, Y) = 0$

3. The joint distribution of two random variables X and Y is given below. Find the marginal distributions of X and Y. Also determine μ_X and μ_Y and the covariance and correlation coefficient of X and Y.

X/Y	-2	-1	4	5
1	0.1	0.2	0.0	0.3
2	0.2	0.1	0.1	0.0

Answers $E(X) = 1.4$ $E(Y) = 1.0$ $E(XY) = 0.9$ Variance of X = 0.24 Variance of Y = 9.6

$Cov(X, Y) = -0.5$ $\rho(X, Y) = -0.3294$

4. The joint distribution of two random variables X and Y is given below. Find the marginal distributions of X and Y. Also determine μ_X and μ_Y and the covariance and correlation coefficient of X and Y.

X/Y	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

Answers : $E(X) = 4$ $E(Y) = 3$ $E(XY) = 12$ $Cov(X, Y) = 0.0$ $\rho(X, Y) = 0.0$

5. The joint distribution of two random variables X and Y is given below. Find the marginal distributions of X and Y. Also determine μ_X and μ_Y and the covariance and correlation coefficient of X and Y.

X/Y	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

Answers: $E(X) = 2$ $E(Y) = 1.05$ $E(XY) = 0.9$ Variance of $X = 1$ Variance of $Y = 9.24$

$\text{Cov}(X,Y) = -1.2$ $\rho(X,Y) = -0.3947$

6. The distributions of two stochastically independent random variables X and Y defined on the sample space are given by the following tables. Find the joint distribution.

X	0	1
P(X)	0.2	0.8

Y	1	2	3
P(Y)	0.1	0.4	0.5

7. The distributions of two random variables X and Y defined on the sample space are given by the following tables. Find the joint distribution. Also find correlation coefficient.

X	1	2
P(X)	0.7	0.3

Y	-2	5	8
P(Y)	0.3	0.5	0.2

8. The joint distribution of two random variables X and Y is defined by the function $P(X,Y) = c(2x+y)$ where X and Y assume the integer values $0,1,2$. Find the marginal distributions of X and Y . Are they independent?

9. Two independent random variables X and Y are such that X takes values $2,5,7$ with the probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ and Y takes values $3,4,5$ with probabilities $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$. Find the joint probability distribution of X and Y . Also calculate the correlation coefficient.

10. A fair coin is tossed thrice. The random variables X and Y are defined as follows: X = 0 or 1 according as head or tail occurs on the first toss. Y = number of heads. Determine the marginal distributions and hence find the correlation coefficient.

Answers: $E(X) = \frac{1}{2}$ $E(Y) = \frac{3}{2}$ $E(XY) = \frac{1}{2}$ Variance of X = $\frac{1}{4}$ Variance of Y = $\frac{3}{4}$

$Cov(X,Y) = -\frac{1}{4}$ $\rho(X,Y) = -0.577$

11. A fair coin is tossed thrice. The random variables X and Y are defined as follows: X = 0 or 1 according as tail or head occurs on the first toss. Y = number of tails. Determine the marginal distributions and hence find the correlation coefficient.

Answers: $E(X) = \frac{1}{2}$ $E(Y) = \frac{3}{2}$ $E(XY) = \frac{1}{2}$ Variance of X = $\frac{1}{4}$ Variance of Y = $\frac{3}{4}$

$Cov(X,Y) = -\frac{1}{4}$ $\rho(X,Y) = -0.577$

**THE DERIVATION OF MEAN AND VARIANCE OF FOUR
DISTRIBUTION IS GIVEN BELOW**

Module - IV

2

①

1. Mean and Variance of Binomial Distribution.

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$\text{Mean} = \sum x P(x) = E(x) = \mu$$

$$= \sum x {}^n C_x p^x q^{n-x}$$

$$= \sum x \frac{n!}{x! (n-x)!} p^x q^{n-x}$$

$$= \sum \frac{n(n-1)!}{(x-1)! [(n-1)-(x-1)]!} p^{x-1} q^{[n-1]-(x-1)}$$

$$= np \sum \frac{(n-1)!}{(x-1)! [(n-1)-(x-1)]!} p^{x-1} q^{(n-1)-(x-1)}$$

$$= np (p+q)^{n-1}$$

$$\therefore p+q=1$$

$$\text{Mean} = np$$

$$\text{Variance} = E(x^2) - [E(x)]^2$$

$$= \sum x^2 p(x) - (np)^2$$

$$= \sum [x(x-1) + x] p(x) - (np)^2$$

$$= \sum x(x-1) p(x) + \sum x p(x) - (np)^2$$

$$= \sum x(x-1) p(x) + np - (np)^2 \rightarrow \text{①}$$

Consider $\sum x(x-1) p(x)$

$$= \sum x(x-1) n \binom{n-1}{x} p^x q^{n-x}$$

$$= \sum x(x-1) \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum \frac{n(n-1)(n-2)!}{(x-2)! [(n-2)-(x-2)]!} p^2 p^{x-2} q^{(n-2)-(x-2)}$$

$$= n(n-1) p^2 \sum \frac{(n-2)!}{(x-2)! [(n-2)-(x-2)]!} p^{x-2} q^{(n-2)-(x-2)}$$

$$= n(n-1) p^2 (p+q)^{n-2}$$

$$= n(n-1) p^2 \text{ using in } \textcircled{1}$$

$$\boxed{\begin{array}{c} \therefore \\ p+q=1 \end{array}}$$

$$\text{Variance} = \sigma^2$$

$$= n(n-1) p^2 + np - (np)^2$$

$$= (n^2 - n) p^2 + np - n^2 p^2$$

$$= \cancel{n^2} p^2 - np^2 + np - \cancel{n^2} p^2$$

$$= np(-p+1)$$

$$\text{Variance} = npq$$

$$\text{SD} = \sqrt{\text{variance}}$$

$$= \sqrt{npq}$$

② Mean and Variance of Poisson Distribution.

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x > 0$$

$$\text{Mean} = E(x) = \mu = \sum x p(x)$$

$$= \sum x \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum \frac{\lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \lambda \sum \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda} \lambda \left\{ 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \right\}$$

$$= e^{-\lambda} \lambda e^{\lambda}$$

$$\text{Mean} = \lambda.$$

$$\text{Variance} = \sigma^2 = E(x^2) - [E(x)]^2$$

$$= \sum x^2 p(x) - \lambda^2$$

$$= \sum [x(x-1) + x] p(x) - \lambda^2$$

$$= \sum x(x-1) p(x) + \sum x p(x) - \lambda^2$$

$$= \sum x(x-1) p(x) + \lambda - \lambda^2$$

→ ①

Consider $\sum x(x-1) p(x)$

(4)

$$= \sum x(x-1) \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum \frac{\lambda^x}{(x-2)!}$$

$$= e^{-\lambda} \lambda^2 \sum \frac{\lambda^{x-2}}{(x-2)!}$$

$$= e^{-\lambda} \lambda^2 \left\{ 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \right\}$$

$$= e^{-\lambda} \lambda^2 e^{\lambda}$$

$$= \lambda^2 \text{ using in } \textcircled{1}$$

$$\text{Variance} = \sigma^2 = \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

$$\text{SD} = \sqrt{\lambda}$$

Q.5n: Prove that the mean and variance of Poisson distribution are same.

① Mean and Variance of Exponential distribution.

$$P(x) = \begin{cases} \lambda e^{-\lambda x} & 0 \leq x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

Mean = $\mu = E(x)$

$$= \int_0^{\infty} x P(x) dx$$

$$= \int_0^{\infty} \lambda x e^{-\lambda x} dx$$

$$\int u v dx = uv - u'v + \dots$$

$$= \lambda \left\{ x \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 1 \left(\frac{e^{-\lambda x}}{\lambda^2} \right) \right\}_0^{\infty}$$

$$= \lambda \left\{ 0 - \left[0 - \frac{1}{\lambda^2} \right] \right\}$$

$$\therefore e^{-\infty} = 0$$

$$= \lambda \frac{1}{\lambda^2} = \frac{1}{\lambda} \Rightarrow \text{mean} = \frac{1}{\lambda}$$

Variance = $\int_0^{\infty} (x - \mu)^2 P(x) dx$

$$= \int_0^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \lambda e^{-\lambda x} dx$$

$$= \lambda \left\{ \left(x - \frac{1}{\lambda}\right)^2 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 2 \left(x - \frac{1}{\lambda}\right) \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 2 \left(\frac{e^{-\lambda x}}{-\lambda^3} \right) \right\}_0^{\infty}$$

$$= \alpha \left\{ 0 - \left[\left(-\frac{1}{\alpha}\right)^2 \left(-\frac{1}{\alpha}\right) - 2 \left(-\frac{1}{\alpha}\right) \left(\frac{1}{\alpha^2}\right) - \frac{2}{\alpha^3} \right] \right\} \quad (6)$$

$$= \alpha \left\{ - \left(-\frac{1}{\alpha^3} + \frac{2}{\alpha^3} - \frac{2}{\alpha^3} \right) \right\}$$

$$= \alpha \left(\frac{1}{\alpha^3} \right) = \frac{1}{\alpha^2}$$

$$\therefore \text{variance} = \frac{1}{\alpha^2}$$

$$SD = \sqrt{\text{variance}} = \frac{1}{\alpha}$$

Note: Prove that the Mean and Standard deviation are same for exponential distribution.

(4) Find Mean and Variance of normal distribution.

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x p(x) dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\text{put } \frac{x-\mu}{\sqrt{2}\sigma} = t \Rightarrow x = \sqrt{2}\sigma t + \mu$$

$$dx = \sqrt{2}\sigma dt$$

$$\begin{aligned} \text{Mean} &= \int_{-\infty}^{\infty} (\sqrt{2}\sigma t + \mu) \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2} \sqrt{2} \sigma dt \quad (2) \\ &= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^{\infty} \sqrt{2}\sigma t e^{-t^2} dt + \int_{-\infty}^{\infty} \mu e^{-t^2} dt \right] \\ &= \frac{1}{\sqrt{\pi}} \left[0 + \mu \int_{-\infty}^{\infty} e^{-t^2} dt \right] \\ &= \frac{1}{\sqrt{\pi}} \mu \sqrt{\pi} = \mu \Rightarrow \boxed{\text{mean} = \mu} \end{aligned}$$

$$\begin{aligned} \text{Variance} &= \int_{-\infty}^{\infty} (x - \mu)^2 p(x) dx \\ &= \int_{-\infty}^{\infty} (x - \mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x - \mu)^2}{2\sigma^2}} dx \end{aligned}$$

$$\text{put } \frac{x - \mu}{\sqrt{2}\sigma} = t \Rightarrow x = \sqrt{2}\sigma t + \mu$$

$$dx = \sqrt{2}\sigma dt$$

$$\begin{aligned} \text{Variance} &= \int_{-\infty}^{\infty} (\sqrt{2}\sigma t)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2} \sqrt{2}\sigma dt \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2\sigma^2 t^2 e^{-t^2} dt \end{aligned}$$

$$= \frac{1}{\sqrt{\pi}} \sigma^2 \int_{-\infty}^{\infty} t^2 e^{-t^2} dt$$

$$= \frac{1}{\sqrt{\pi}} \sigma^2 \sqrt{\pi} = \sigma^2$$

$$\text{Variance} = \sigma^2 \quad \boxed{\therefore \text{SD} = \sigma}$$