MODULE - 2

MATHEMATICAL MODELS

LESSON STRUCTURE:

- **2.1. Modeling of Control Systems**
- **2.2. Modeling of Mechanical Systems**
- **2.3. Mathematical Modeling of Electrical System**
- **2.4. Force Voltage Analogy**
- **2.5. Force Current Analogy**
- **2.6. Transfer Functions definition**
- **2.7. Block Diagram:**
- **2.8. Signal Flow Graphs**
- **2.9. Mason's Gain Formula**

OBJECTIVES:

- To develop mathematical model for the mechanical, electrical, servo mechanism and hydraulic systems.
- To teach students the concepts of block diagrams and transfer functions.
- To teach students the concepts of Signal flow graph.

2.1. Modeling of Control Systems:

The first step in the design and the analysis of control system is to build physical and mathematical models. A control system being a collection of several physical systems (sub systems) which may be of mechanical, electrical electronic, thermal, hydraulic or pneumatic type. No physical system can be represented in its full intricacies. Idealizing assumptions are always made for the purpose of analysis and synthesis. An idealized representation of physical system is called a Physical Model.

Control systems being dynamic systems in nature require a quantitative mathematical description of the system for analysis. ***This process of obtaining the desired mathematical description of the system is called Mathematical Modeling***.

In Unit 1, we have learnt the basic concepts of control systems such as open loop and feedback control systems, different types of Control systems like regulator systems, follow-up systems and servo mechanisms. We have also discussed a few simple applications. In this chapter we learn the concepts of modeling.

The requirements demanded by every control system are many and depend on the system under consideration. Major requirements are 1) Stability 2) Accuracy and 3) Speed of Response. An ideal control system would be stable, would provide absolute accuracy (maintain zero error despite disturbances) and would respond instantaneously to a change in the reference variable. Such a system cannot, of course, be produced. However, study of automatic control system theory would provide the insight necessary to make the most effective compromises so that the engineer can design the best possible system. One of the important steps in the study of control systems is modeling. Following section presents modeling aspects of various systems that find application in control engineering.

The basic models of dynamic physical systems are differential equations obtained by the application of appropriate laws of nature. Having obtained the differential equations and where possible the numerical values of parameters, one can proceed with the analysis. When the mathematical model of a physical system is solved for various input conditions, the results represent the dynamic response of the system. The mathematical model of a system is linear, if it obeys the principle of ***superposition and homogeneity***.

A mathematical model is ***linear***, if the differential equation describing it has coefficients, which are either functions of the independent variable or are constants. If the coefficients of the describing differential equations are functions of time (the independent variable), then the mathematical model is ***linear time-varying***. On the other hand, if the coefficients of the describing differential equations are constants, the model is ***linear time-invariant***. Powerful mathematical tools like the Fourier and Laplace transformations are available for use in linear systems. Unfortunately no physical system in nature is perfectly linear. Therefore certain assumptions must always be made to get a linear model.

Usually control systems are complex. As a first approximation a simplified model is built to get a general feeling for the solution. However, improved model which can give better accuracy can then be obtained for a complete analysis. Compromise has to be made between simplicity of the model and accuracy. It is difficult to consider all the details for mathematical analysis. Only most important features are considered to predict behavior of the system under specified conditions. A more complete model may be then built for complete analysis.

2.2. Modeling of Mechanical Systems:

Mechanical systems can be idealized as spring- mass-damper systems and the governing differential equations can be obtained on the basis of Newton's second law of motion, which states that

$$F = ma\text{: for rectilinear motion}$$

where F: Force, m: mass and a: acceleration (with consistent units)

$T = I\ \alpha$: or $J\alpha$ for rotary motion

where T: Torque, I or J: moment of inertia and α: angular acceleration (with consistent units)

Mass / inertia and the springs are the energy storage elements where in energy can be stored and retrieved without loss and hence referred as conservative elements. Damper represents the energy loss (energy absorption) in the system and hence is referred as dissipative element. Depending upon the choice of variables and the coordinate system, a given physical model may lead to different mathematical models. The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time defines the degrees of freedom (DOF) of the system. A large number of practical systems can be described using a finite number of degrees of freedom and are referred as discrete or lumped parameter systems. Some systems, especially those involving continuous elastic members, have an infinite number of degrees of freedom and are referred as continuous or distributed systems. Most of the time, continuous systems are approximated as discrete systems, and solutions are obtained in a simpler manner. Although treatment of a system as continuous gives exact results, the analysis methods available for dealing with continuous systems are limited to a narrow selection of problems. Hence most of the practical systems are studied by treating them as finite lumped masses, springs and dampers. In general, more accurate results are obtained by increasing the number of masses, springs and dampers-that is, by increasing the number of degrees of freedom.

Mechanical systems can be of two types:

1) Translation Systems
2) Rotational Systems.

The variables that described the motion are displacement, velocity and acceleration.

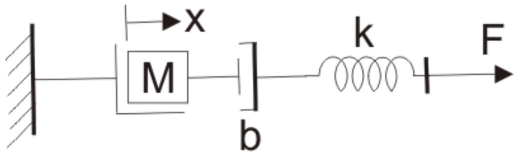
And also we have three parameters-

- Mass which is represented by 'M'.
- Coefficient of viscous friction which is represented by 'B'.
- Spring constant which is represented by 'K'.

In rotational mechanical type of systems we have three variables-

- Torque which is represented by 'T'.
- Angular velocity which is represented by 'ω'
- Angular displacement represented by 'θ'

Now let us consider the linear displacement mechanical system which is shown below-



spring mass mechanical system

We have already marked various variables in the diagram itself. We have x is the displacement as shown in the diagram. From the Newton's second law of motion, we can write force as

$$F_1 = M\frac{dx^2}{d^2y} + B\frac{dt}{dy} + Kx$$
$$F_2 = B\frac{dt}{dx}$$
$$F_3 = Kx$$

From the diagram we can see that the,

$$F = F_1+F_2+F_3$$

On substituting the values of F_1, F_2 and F_3 in the above equation and taking the Laplace transform we have the transfer function as,

$$T = \frac{1}{Ms^2 + Bs + K}$$

2.3. Mathematical Modeling of Electrical System:

In electrical type of systems we have three variables -

- Voltage which is represented by 'V'.
- Current which is represented by 'I'.
- Charge which is represented by 'Q'.

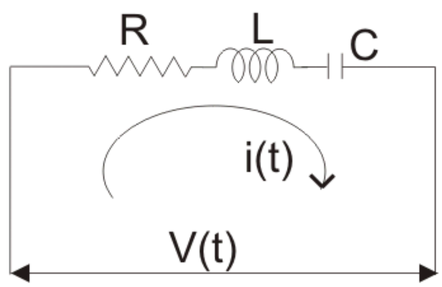
And also we have three parameters which are active and passive elements –

- Resistance which is represented by 'R'.
- Capacitance which is represented by 'C'.
- Inductance which is represented by 'L'.

Now we are in condition to derive analogy between electrical and mechanical types of systems. There are two types of analogies and they are written below:

2.4. Force Voltage Analogy :

In order to understand this type of analogy, let us consider a circuit which consists of series combination of resistor, inductor and capacitor.



A voltage V is connected in series with these elements as shown in the circuit diagram. Now from the circuit diagram and with the help of KVL equation we write the expression for voltage in terms of charge, resistance, capacitor and inductor as,

$$V = L\frac{dt^2}{d^2q} + R\frac{dt}{dq} + \frac{q}{C}$$

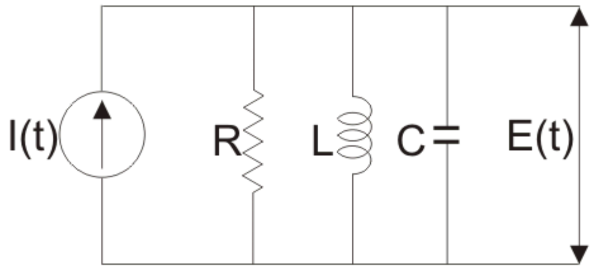
Now comparing the above with that we have derived for the mechanical system we find that-

1. Mass (M) is analogous to inductance (L).
2. Force is analogous to voltage V.
3. Displacement (x) is analogous to charge (Q).
4. Coefficient of friction (B) is analogous to resistance R and
5. Spring constant is analogous to inverse of the capacitor (C).

This analogy is known as force voltage analogy.

2.5. Force Current Analogy :

In order to understand this type of analogy, let us consider a circuit which consists of parallel combination of resistor, inductor and capacitor.



A voltage E is connected in parallel with these elements as shown in the circuit diagram. Now from the circuit diagram and with the help of KCL equation we write the expression for current in terms of flux, resistance, capacitor and inductor as,

$$I = C\frac{dt^2}{d^2\psi} + \frac{1}{R}\frac{dt}{d\psi} + \frac{\psi}{L}$$

Now comparing the above with that we have derived for the mechanical system we find that,

1. Mass (M) is analogous to Capacitor (C).
2. Force is analogous to current I.
3. Displacement (x) is analogous to flux (ψ).
4. Coefficient of friction (B) is analogous to resistance 1/ R and
5. Spring constant K is analogous to inverse of the inductor (L).

This analogy is known as force current analogy.

2.6. Transfer Functions definition

The transfer function of a control system is defined as the ration of the Laplace transform of the output variable to Laplace transform of the input variable assuming all initial conditions to be zero.

$$G(s) = \frac{C(s)}{R(s)} \Rightarrow R(s).G(s) = C(s)$$

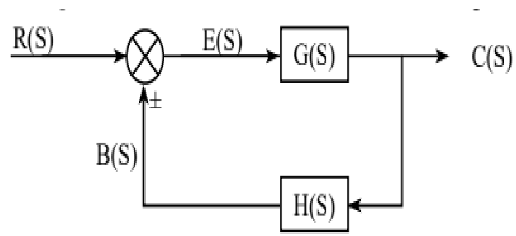
2.7.Block Diagram:

A control system may consist of a number of components. In order to show the functions performed by each component in control engineering, we commonly use a diagram called the **Block Diagram**.

A block diagram of a system is a pictorial representation of the function performed by each component and of the flow of signals. Such a diagram depicts the inter-relationships which exists between the various components. A block diagram has the advantage of indicating more realistically the signal flows of the actual system.

In a block diagram all system variables are linked to each other through functional blocks. The ―Functional Block or simply ―Block is a symbol for the mathematical operation on the input signal to the block which produces the output. The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of flow of signals. Note that signal can pass only in the direction of arrows. Thus a block diagram of a control system explicitly shows a unilateral property.

Block diagram of a closed loop system.



The output C(s) is fed back to the summing point, where it is compared with reference input R(s). The closed loop nature is indicated in fig1.3. Any linear system may be represented by a block diagram consisting of blocks, summing points and branch points. A branch is the point from which the output signal from a block diagram goes concurrently to other blocks or summing points.

When the output is fed back to the summing point for comparison with the input, it is necessary to convert the form of output signal to that of he input signal. This conversion is followed by the feedback element whose transfer function is H(s) as shown in fig 1.4. Another important role of the feedback element is to modify the output before it is compared with the input.

The ratio of the feedback signal B(s) to the actuating error signal E(s) is called the open loop transfer function.

Open loop transfer function = B(s)/E(s) = G(s)H(s)

The ratio of the output C(s) to the actuating error signal E(s) is called the feed forward transfer function.

Feed forward transfer function = C(s)/E(s) = G(s)

If the feedback transfer function is unity, then the open loop and feed forward transfer function are the same. For the system shown in Fig1.4, the output C(s) and input R(s) are related as follows.

$$C(s) = G(s)\,E(s)$$

$$E(s) = R(s) - B(s)$$

$$= R(s) - H(s)\,C(s)$$

But $B(s) = H(s)\,C(s)$

Eliminating E(s) from these equations

$$C(s) = G(s)\,[R(s) - H(s)\,C(s)]$$

$$C(s) + G(s)\,[H(s)\,C(s)] = G(s)\,R(s)$$

$$C(s)[1 + G(s)H(s)] = G(s)R(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)}$$

C(s)/R(s) is called the closed loop transfer function.

The output of the closed loop system clearly depends on both the closed loop transfer function and the nature of the input. If the feedback signal is positive, then

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)\,H(s)}$$

2.8.SIGNAL FLOW GRAPHS

An alternate to block diagram is the signal flow graph due to S. J. Mason. A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. Each signal flow graph consists of a network in which nodes are connected by directed branches. Each node represents a system variable, and each branch acts as a signal multiplier. The signal flows in the direction indicated by the arrow.

Definitions:

Node: A node is a point representing a variable or signal.
Branch: A branch is a directed line segment joining two nodes.
Transmittance: It is the gain between two nodes.
Input node: A node that has only outgoing branches. It is also, called as source and corresponds

to independent variable.

Output node: A node that has only incoming branches. This is also called as sink and Corresponds to dependent variable.

Mixed node: A node that has incoming and outgoing branches.

Path: A path is a traversal of connected branches in the direction of branch arrow.

Loop: A loop is a closed path.

Self loop: It is a feedback loop consisting of single branch.

Loop gain: The loop gain is the product of branch transmittances of the loop.

Non-touching loops: Loops that do not possess a common node.

Forward path: A path from source to sink without traversing an node more than once.

Feedback path: A path which originates and terminates at the same node.

Forward path gain: Product of branch transmittances of a forward path.

Properties of Signal Flow Graphs:

1. Signal flow applies only to linear systems.
2. The equations based on which a signal flow graph is drawn must be algebraic equations in the form of effects as a function of causes. Nodes are used to represent variables. Normally the nodes are arranged left to right, following a succession of causes and effects through the system.
3. Signals travel along the branches only in the direction described by the arrows of the branches.
4. The branch directing from node X_k to X_j represents dependence of the variable Xj on X_k but not the reverse.
5. The signal traveling along the branch X_k and X_j is multiplied by branch gain a_{kj} and signal $a_{kj}X_k$ is delivered at node X_j.

Guidelines to Construct the Signal Flow Graphs:

The signal flow graph of a system is constructed from its describing equations, or by direct reference to block diagram of the system. Each variable of the block diagram becomes a node and each block becomes a branch. The general procedure is

1. Arrange the input to output nodes from left to right.
2. Connect the nodes by appropriate branches.
3. If the desired output node has outgoing branches, add a dummy node and a unity gain branch.
4. Rearrange the nodes and/or loops in the graph to achieve pictorial clarity.

2.9. Mason's Gain Formula:

The relationship between an input variable and an output variable of a signal flow graph is given by the net gain between input and output nodes and is known as overall gain of the system. Masons gain formula is used to obtain the overall gain (transfer function) of signal flow graphs.

Gain P is given by

$$P = \frac{1}{\Delta} \sum_k P_k \Delta_k$$

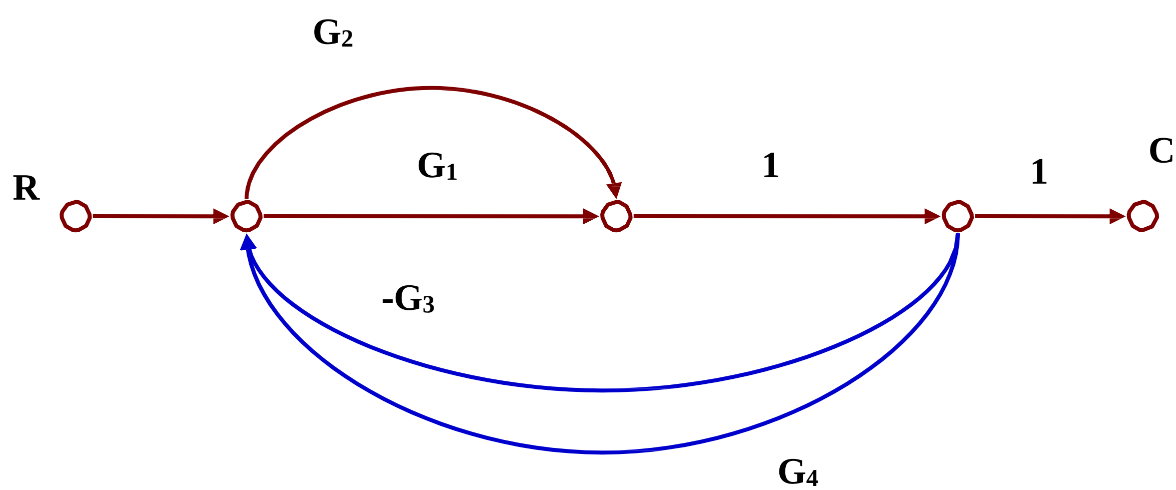
Where, P_k is gain of k^{th} forward path,
Δ is determinant of graph

Δ=1-(sum of all individual loop gains) + (sum of gain products of all possible combinations of two non touching loops – sum of gain products of all possible combination of three non touching loops) + ···

Δ_k is cofactor of k^{th} forward path determinant of graph with loops touching k^{th} forward path. It isobtained from Δ by removing the loops touching the path P_k.

Example 1

Obtain the transfer function of C/R of the system whose signal flow graph is shown in Figure



Solution:

There are two forward paths:

Gain of path 1 : P_1=G_1

Gain of path 2 : P_2=G_2

There are four loops with loop gains:

L_1=-G_1G_3, L_2=G_1G_4, L_3= -G_2G_3, L_4= G_2G_4

There are no non-touching loops.

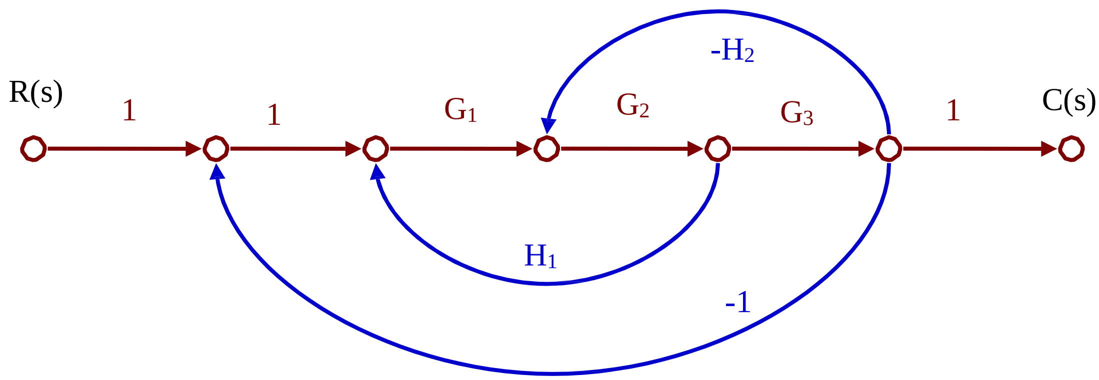
$\Delta = 1+G_1G_3-G_1G_4+G_2G_3-G_2G_4$

Forward paths 1 and 2 touch all the loops. Therefore, Δ_1= 1, Δ_2= 1

The transfer function T = $\frac{C(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1 + G_2}{1+G_1G_3 - G_1G_4 + G_2G_3 - G_2G_4}$

Example 2

Obtain the transfer function of C(s)/R(s) of the system whose signal flow graph is shown in Figure



There is one forward path, whose gain is: $P_1=G_1G_2G_3$

There are three loops with loop gains:

L_1=-$G_1G_2H_1$,

L2=$G_2G_3H_2$,

L3= -$G_1G_2G_3$

There are no non-touching loops.

$\Delta = 1-G_1G_2H_1+G_2G_3H_2+G_1G_2G_3$

Forward path 1 touches all the loops. Therefore, Δ_1= 1.

The transfer function T = $\frac{C(s)}{R(s)} = \frac{P_1\Delta_1}{\Delta}$

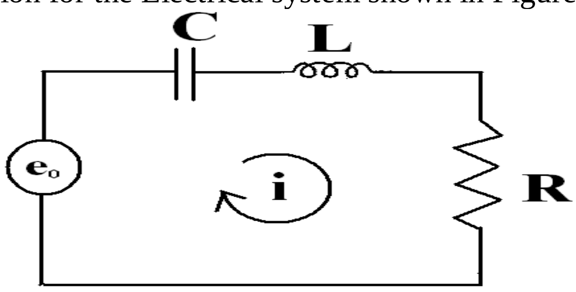
OUTCOMES:

At the end of the unit, the students are able to:

- Mathematical modeling of mechanical, electrical, servo mechanism and hydraulic systems.
- To find Transfer function of a system.
- Calculate the gain of the system using block diagram and signal flow graph and to illustrate the response of systems.

SELF-TEST QUESTIONS:

1. What mathematical model permits easy interconnection of physical systems?
2. Define the transfer function.
3. What are the component parts of the mechanical constants of a motor's transfer function?
4. Derive the transfer function of a Spring - Mass-Damper – system.
5. Differentiate between FI and FV analogy.
6. Obtain Transfer function of Armature controlled DC motor.
7. Derive transfer function for the Electrical system shown in Figure below.



8. Differentiate between Block diagram and Signal flow graph techniques.
9. Explain the rules for constructing Signal flow graph.
10. Reduce the block diagram shown in Figure 1, to its simplest possible form and find its closed loop transfer function.

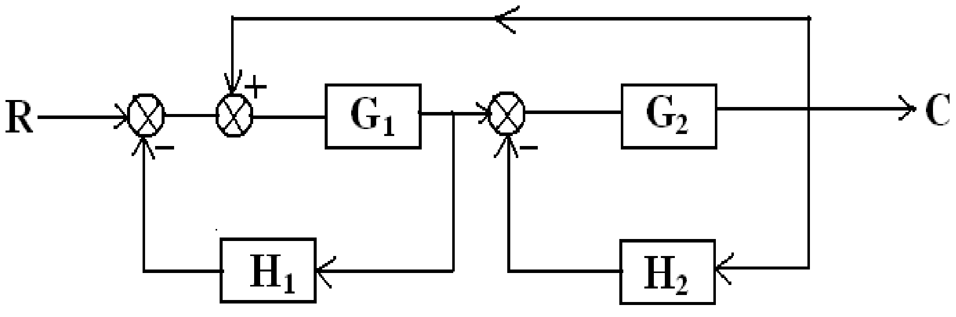


Figure 1

11. Find **C(S)/R(S)** for the following system using Mason's gain rule shown in figure 2.

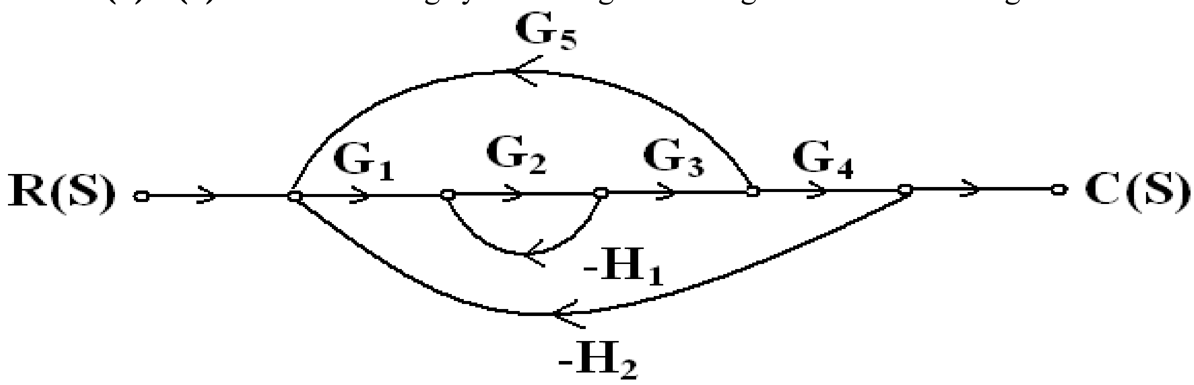


Figure 2

FURTHER READING:

1. **Control engineering,** Swarnakiran S, Sunstar publisher, 2018.
2. **Feedback Control System**, Schaum's series. 2001.