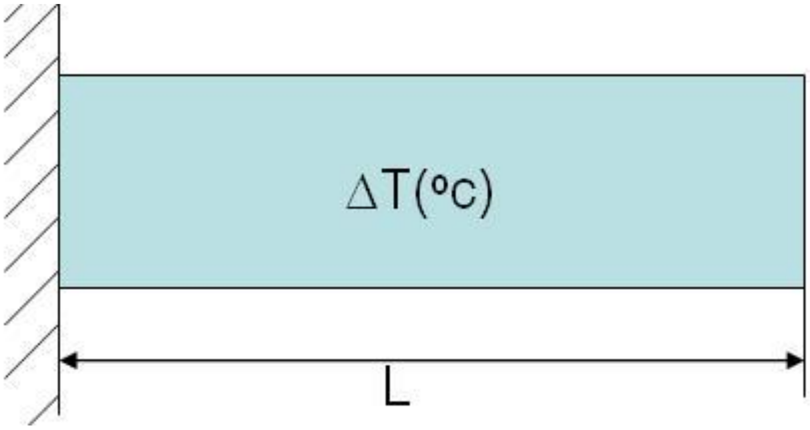
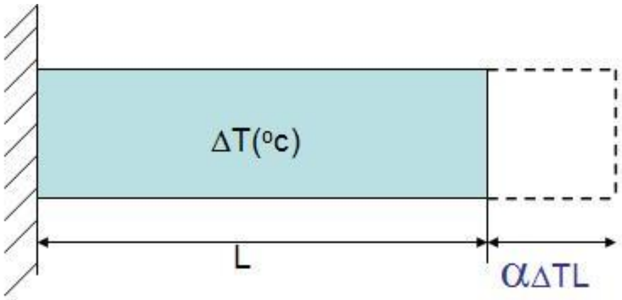
Module 4

Temperature effect on 1D bar element

Lets us consider a bar of length L fixed at one end whose temperature is increased to ΔT as shown.



Because of this increase in temperature stress induced are called as thermal stress and the bar gets expands by a amount equal to αΔTL as shown. The resulting strain is called as thermal strain or initial strain

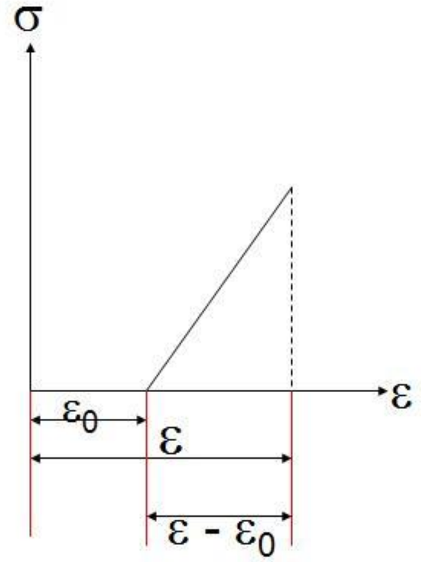


α = coefficient of thermal expansion

$$\varepsilon_o = \frac{\alpha \Delta T L}{L} = \alpha \Delta T$$

Thermal strain (initial strain)

In the presence of this initial strain variation of stress strain graph is as shown below



Hooke's law

$$\frac{\text{Stress}}{\text{Strain}} = \frac{\sigma}{\varepsilon - \varepsilon_0} = E$$

$$\sigma = (\varepsilon - \varepsilon_0)\, E$$

We know that

Strain energy in a bar

$$U = \tfrac{1}{2} \int \sigma^T \varepsilon \, dv$$

For an element

$$U = \tfrac{1}{2} \int_e \sigma^T \varepsilon \, A \, dx$$

Therefore

$$U = \tfrac{1}{2} \int_e E \, (\varepsilon - \varepsilon_0)^T (\varepsilon - \varepsilon_0) \, A \, dx$$

$$U = \tfrac{1}{2} \int_e E \, (Bq - \varepsilon_0)^T (Bq - \varepsilon_0) \, A \, dx$$

$$U = \frac{1}{2} \int_e E\,(Bq - \varepsilon_0)^T\,(Bq - \varepsilon_0)\; A\; dx$$

But $dx/d\xi = L_e/2$

$$U = \frac{1}{2}\, EA \int_e (Bq - \varepsilon_0)^T\,(Bq - \varepsilon_0)\; Le/2\; d\xi$$

$$U = \frac{1}{2}\, EA/2 \int_e (Bq - \varepsilon_0)^T\,(Bq - \varepsilon_0)\; Le\; d\xi$$

$$U = \frac{1}{2}\, EA/2 \int_e (B^T q^T - \varepsilon_0)\,(Bq - \varepsilon_0)\; Le\; d\xi$$

$$U = \frac{1}{2}\, Le\; EA/2 \int_e [B^T q^T B q - B^T q^T \varepsilon_0 - Bq\varepsilon_0 + \varepsilon_0^2]\; d\xi$$

$$U = \frac{1}{2}\, Le\; EA/2 \int_e [B^T q^T B q - \varepsilon_0\,(B^T q^T + Bq) + \varepsilon_0^2]\; d\xi$$

Therefore

$$U = \frac{1}{2}\, Le\; EA/2 \int_e [B^T q^T B q - \varepsilon_0\, 2B^T q^T + \varepsilon_0^2]\; d\xi$$

Integrating individual terms

Stiffness matrix

$$U = \frac{1}{2} q^T EA \frac{Le}{2} \int_e [B^T B\, d\xi\,]\, q$$

Thermal load vector

$$- \frac{1}{2} q^T EA \frac{Le}{2}\, \varepsilon_0 \int_e 2B^T\, d\xi$$

$$+ \frac{1}{2} EA \frac{Le}{2} \cancelto{0}{\int_e \varepsilon_0^2\, d\xi}$$

Extremizing the potential energy first term yields stiffness matrix, second term results in thermal load vector and last term eliminates that do not contain displacement filed

Thermal load vector

From the above expression taking the thermal load vector lets derive what is the effect of thermal load.

$$\theta_e = \frac{1}{\cancel{2}} \quad \frac{EA\,Le}{2}\,\varepsilon_0 \int_e \cancel{2}B^T\,d\xi$$

$$= \frac{1}{} \quad \frac{EA\,Le}{2}\,\varepsilon_0 \int_e \; B^T\,d\xi$$

We know that $B^T = \frac{1}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\theta_e = \frac{EA}{2}\varepsilon_0 \int_{-1}^{1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} d\xi$$

$$= \frac{EA}{2}\varepsilon_0 \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$= EA\ \varepsilon_0. \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\theta = EA\alpha_{\Delta T} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Stress component because of thermal load

$$\sigma = (\varepsilon - \varepsilon_0)\, E$$

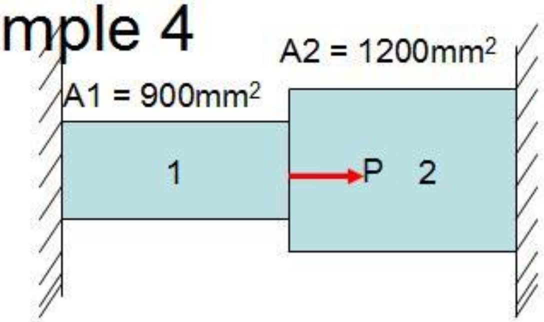
We know $\varepsilon = Bq$ and $\varepsilon_o = \alpha \Delta T$ substituting these in above equation we get

$$= (Bq - \alpha\Delta T)\, E$$

$$= E\, Bq - E\, \alpha\Delta T$$

$$\sigma = E\, \frac{1}{L} [-1 \quad 1] q - E\, \alpha\Delta T$$

Example 4



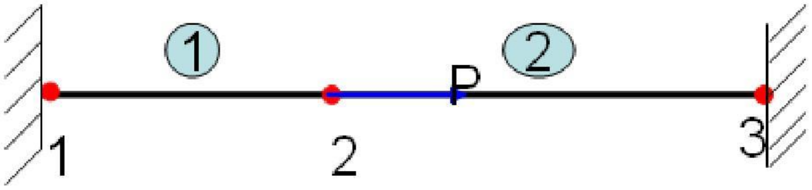
$\alpha_1 = 23 \text{ X } 10^{-6} \text{ Per } {}^0C$ $\alpha_2 = 11.7 \text{ X } 10^{-6} \text{ Per } {}^0C$

$E_1 = 70 \text{ x } 10^9 \text{ N/m}^2$ $E_2 = 200 \text{ X } 10^9 \text{ N/m}^2$

$L_1 = 200\text{mm}$ $L_2 = 300\text{mm}$

P = 300 KN is applied at 20^0c ,the temperature is then raised to 60^0c

Solution:



$$K_1 = \frac{A_1E_1}{L_1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{900\text{x}70\text{x } 10^3}{200}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 10^3 \begin{array}{cc|c} 1 & 2 & \\ 315 & -315 & 1 \\ -315 & 315 & 2 \end{array}$$

$$K_2 = \frac{A_2E_2}{L_2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 10^3 \begin{array}{cc|c} 2 & 3 & \\ 800 & -800 & 2 \\ -800 & 800 & 3 \end{array}$$

Global stiffness matrix:

$$K=\begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \left(\begin{array}{ccc} 315 & -315 & 0 \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{array}\right)10^3 & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

Thermal load vector:

We have the expression of thermal load vector given by

$$\theta = EA\alpha\Delta T\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Element 1

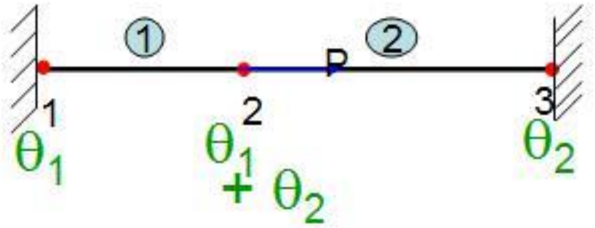
$$\theta_1 = 70 \text{ X } 10^3 \text{ X } 900 \text{ X } 23 \text{ X } 10^{-6} \text{ X } 40 \begin{array}{cc} \begin{pmatrix} -1 \\ 1 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$$\theta_1 = 10^3 \begin{array}{cc} \begin{pmatrix} -57.96 \\ 57.96 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

Similarly calculate thermal load distribution for second element

$$\theta_2 = 10^3 \begin{array}{cc} \begin{pmatrix} -112.32 \\ 112.32 \end{pmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

Global load vector:



$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ P + \theta_2 + \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} -57.96 \\ 245.64 \\ 112.32 \end{pmatrix} 10^3$$

From the equation KQ=F we have

$$\begin{pmatrix} 315 & -315 & 0 \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{pmatrix} 10^3 \begin{pmatrix} Q1 \\ Q2 \\ Q3 \end{pmatrix} = \begin{pmatrix} -57.96 \\ 245.64 \\ 112.32 \end{pmatrix} \begin{matrix} \\ 10^3 -(-315 \times 10^3)Q1-(-8 \times 10^5)Q3 \\ -(0)Q1 \end{matrix}$$

After applying elimination method and solving the matrix we have Q2= 0.22mm

Stress in each element:

For element 1

$$\sigma_1 = E_1 \frac{1}{L_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{pmatrix} Q1 \\ Q2 \end{pmatrix} - E_1 \alpha_1 \Delta T$$

$$= 12.60\text{MPa}$$

For element 2

$$\sigma_2 = E_2 \frac{1}{L_2} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{pmatrix} Q2 \\ Q3 \end{pmatrix} - E_2 \alpha_2 \Delta T$$

$$= -240.27\text{MPa}$$