Module 4

TORSION OF SHAFTS

Objectives:

Explain the structural behavior of members subjected to torque, Calculate twist and stress induced in shafts subjected to bending and torsion. & Understand the concept of stability and derive crippling loads for columns

Learning Structure

- 4.1 Bending Moment
- 4.2 ASSUMPTIONS IN TORSION THEORY
- 4.3 Problems
- 4.4 Columns and Struts:
- 4.5 SLENDERNESS RATIO
- 4.6 EFFECTIVE LENGTH OF COLUMN
- .7 Euler's Theorem
- Outcomes
- Further Reading

4.1 Bending Moment

The moment applied in a vertical plane containing the longitudinal axis is resisted by longitudinal tensile and compressive stresses of varying intensities across the depth of bea m and are called as bending stresses. The moment applied is called Bending Moment.

4.1.1 Torsional Moment

The moment applied in a vertical plane perpendicular to the longitudinal axis i.e., in the plane of the cross section of the member, it causes twisting of layers which will be resisted by the shear stresses. The moment applied is called Torsion Moment or Torsional Moment. Torsion is useful form of transmitting power and its application is seen in screws and shafts.

4.2 ASSUMPTIONS IN TORSION THEORY

1. Material is homogenous and isotropic
2. Plane section remain plane before and after twisting i.e., no warpage of planes.
3. Twist along the shaft is uniform.
4. Radii which are straight before twisting remain straight after twisting.
5. Stresses are within the proportional limit.

4.2.1 DERIVATION OF TORSIONAL EQUATION:

Torsional Rigidity

$$\text{We have} \qquad \theta = \frac{TL}{CI_P}$$

As product (CI_P) is increased deformation q reduces. This product gives the strength of the section to resist torque and is called Torsional rigidity.

Polar Modulus : (Z_P)

$$\text{We have} \quad \frac{T}{I_p} = \frac{f}{r}$$

Maximum shear stress occurs at surface

$$T = f_s \cdot \frac{I_p}{R}$$

$$T = f_s \cdot Z_p$$

Where Z_P is called polar modulus $\quad Z_p = \frac{I_p}{R}$

POWER TRANSMITTED BY SHAFT

Power transmitted = Torsional moment x Angle through which the torsional moment rotates / unit tank

If the shaft rotates with 'N' rpm

$$= T\left(\frac{N \cdot 2\pi}{60'}\right)$$

$$\text{Power transmitted} = \frac{2\pi NT}{60} \text{ N.m/sec}$$

$$\text{Power transmitted in kw} = \frac{2\pi NT}{60 \times 1000} = \frac{\pi NT}{30{,}000}$$

Note:

N is in rpm and T is in N-m

4.3 Problems:

1. **Find the maximum shear stress induced in a solid circular shaft of diameter 200 mm when the shaft transmits 190 kW power at 200 rpm**

Given data: Power transmitted, P = 190 kW, $I_p = 1.57 \times 10^8 \text{ mm}^4$

speed N = 200 rpm and diameter of shaft = 200 mm.

Substituting all the values $f_s = 5.78 N/mm^2$.

2. **A solid shaft of mild steel 200 mm in diameter is to be replaced by hollow shaft of allowable shear stress is 22% greater. If the power to be transmitted is to be increased by 20% and the speed of rotation increased by 6%, determine the maximum internal diameter of the hollow shaft. The external diameter of the hollow shaft is to be 200 mm**.

Solution: Given that:

Diameter of solid shaft	$d = 200$ mm
For hollow shaft diameter,	$d_0 = 200$ mm
Shear stress;	$t_H = 1.22\ t_s$
Power transmitted;	$P_H = 1.20\ P_s$
Speed	$N_H = 1.06\ N_s$

As the power transmitted by hollow shaft

$$P_H = 1.20\ P_s$$

$$(2\pi.N_H.T_H)/60 = (2\pi.N_s.T_s)/60 \times 1.20$$

$$N_H.T_H = 1.20\ N_s.T_s$$

$$1.06\ N_s.T_H = 1.20\ N_sT_s$$

$$1.06/1.20\ T_H = T_s$$

$$1.06/1.20 \times \pi/16\ t_H\ [(d_0)^4 - (d_i)^4/d_0] = \pi/16\ t_s.[d]^3$$

$$1.06/1.20 \times 1.22\ t_s\ [(200)^4 - (d_i)^4/200] = t_s \times [200]^3$$

$$d_i = 104 \text{ mm}$$

3. **A solid shaft is subjected to a maximum torque of 1.5 MN.cm Estimate the diameter for the shaft, if the allowable shearing stress and the twist are limited to 1 kN/cm^2 and 1o respectively for 200 cm length of shaft. Take G = 80 × 10^5 N/cm^2**

Solution: Since we have

$$T/I_p = f_s/r = C.\theta/L$$

$$f_s = T.I_p\ r = 1.5 \times 10^6 / \theta/32.d^4 . d/2$$

$$1 \times 10^3 * 2\pi /1.5 \times 10^6 * 32 = 1/d^3$$

$$d = 19.69 \text{ cm}$$

$$\theta = T.L / C.I_p$$

$$1.5 \times 10^6 * 2\pi / 1.5 \times 10^6 * 32 = 1 / d^3$$

$$d = 19.69 \text{ cm}$$

$$\theta = T.L / C.Ip$$

$$1.5 \times 10^6 * 200\ d/80 * 10^5 * \pi/32\ d^4 = \pi/180$$

$d^3 = 1.5 \times 10^6 * 180 * 200 * 32 / (80 * 10^5 * \pi * \pi)$

$d = 27.97$ cm

4. **A hollow circular shaft of 20 mm thickness transmits 300 kW power at 200 r.p.m. Determine the external diameter of the shaft if the shear strain due to torsion is not to exceed 0.00086. Take modulus of rigidity = 0.8×10^5 N/mm^2.**

Solution: Let d_i = inner diameter of circular shaft

d_0 = outer diameter of circular shaft

Then $d_0 = d_i + 2t$ where t = thickness

$d_0 = d_i + 2 * 20$

$d_0 = d_i + 40$

$d_i = d_0 - 40$

Since we have

Power transmitted = 2π NT/60

$300{,}000 = 2\pi * 200 * T / 60$

→ $T = 14323900$ N mm

Also, we have $C = f_s/y$

→ $0.8 * 10^5 = f_s / 0.00086$

→ $f_s = 68.8$ N/mm^2

Now $T = \pi/16.\ f_s.(d_0^4 - d_i^4 / d_0)$

$14323900 = f_s /16 * 68.8\ (d_0^4 - (d_0 - 40)^4 / d_0)$

$1060334.6\ d_0 = d_0^4 - (d_0 - 40)^4$

$= (d_0^2 - d_0^2 + 80d_0 - 1600)*(d_0^2 + d_0^2 - 80d_0 + 1600)$

$= (80d_0 - 1600)\ (2d_0^2 - 80d_0 + 1600)$

$= 80\ (d_0 - 2_0) * 2 * (d_0^2 - 40\ d_0 + 800)$

$= 160\ (d_0^3 - 40d_0^2 + 800\ d_0 - 20\ d_0^2 + 800\ d_0 - 16000)$

→ $1060334.6\ d_0 / 160 = d_0^3 - 60d_0^2 + 1600d_0 - 16000$

→ $6627\ d_0 = d_0^3 - 60d_0^2 + 1600\ d_0 - 1600$

→ $d_0^3 - 60d_0^2 + 1600d_0 - 6627\ d_0 - 16000 = 0$

→ $d_0^3 - 60d_0^2 - 5027\ d_0 - 16000 = 0$

Using trial and error method to solve the above equation for d_0, we get $d_0 = 107.5$ mm.

Elastic Stability of Columns

4.4 Columns and Struts:

Columns and struts are structural members subjected to compressive forces. Theses members are often subjected to axial forces, although they may be loaded eccentrically. The lengths of these members are large compared to their lateral dimensions. In general vertical compressive members called columns and inclined compressive members are called struts.

4.4.1 CLASSIFICATION OF COLUMNS:

Columns are generally classified in to three general types. The distinction between types of columns is not well, but a generally accepted measure is based on the slenderness ratio (l_e/r_{min}).

4.4.1 .1 Short Column :

A short column essentially fails by crushing and not by buckling. A column is said to be short, if $l_e /b \leq 15$ or $l_e /r_{min} \leq 50$, where l_e = effective length, b = least lateral dimension and r_{min}= minimum radius of gyration.

4.4.1 .2 Long Column :

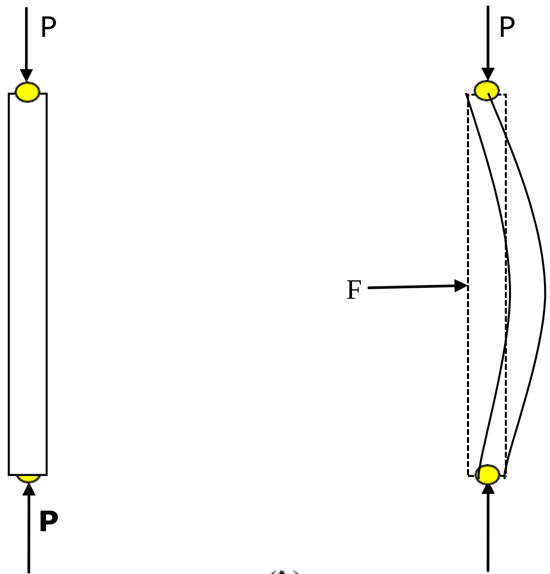
A long column essentially fails by buckling and not by crushing. In long columns, the stress at failure is less than the yield stress. A column is said to be long $l_e/b > 15$ or $l_e /r_{min} > 50$.

4.4.1 .3 Intermediate Column :

An intermediate column is one which fails by a combination of crushing and buckling.

4.4.1.4 Elastic Stability of Column

Consider a long column subjected to an axial load P as shown in figure. The column deflects laterally when a small test load F is applied in lateral direction. If the axial load is small, the column regains its stable position when the test load is removed. At a certain value of the axial load, the column fails to regain its stable position even after the removal of the test load. The column is then said to have failed by buckling and the corresponding axial load is called Critical Load or failure Load or Crippling Load



4.5 SLENDERNESS RATIO (λ)

Slenderness ratio is defined as the ratio of effective length (l_e) of the column to the minimum radius of gyration (r_{min}) of the cross section.

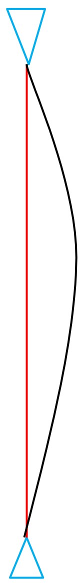
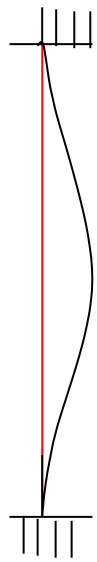
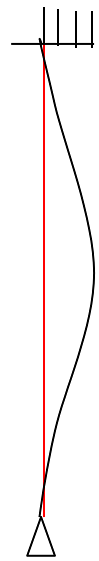
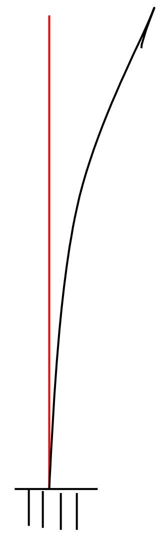
$$\lambda = \frac{l_e}{r_{min}}$$

Since an axially loaded column tends to buckle about the axis of minimum moment of inertia (I_{min}), the minimum radius of gyration is used to calculate slenderness ratio.

Further, —, where A is the cross sectional area of column.

4.6 EFFECTIVE LENGTH OF COLUMN (*l*e)

Effective length is the length of an imaginary column with both ends hinged and whose critical load is the same as the column with given end conditions. It should be noted that the material and geometric properties should be the same in the above columns. The effective length of a column depends on its end condition. Following are the effective lengths for some standard cases.

Both endsare hinged	Both ends arefixed	One end fixed and other end hinged	One end fixed and other end is free
			
Effective Length L_e = L	Effective Length L_e = –	Effective Length L_e = —=	Effective Length L_e = 2L

4.7 Euler's Theorem

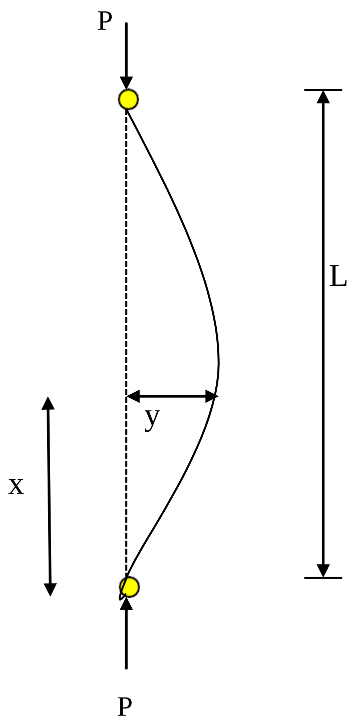
Theoretical analysis of the critical load for long columns was made by the great Swiss mathematician Leonard Euler (pronounced as Oiler). The assumptions made in the analysis are as follows:

- The column is long and fails by buckling.
- The column is axially loaded.
- The column is perfectly straight and the cross sections are uniform (prismatic).
- The column is initially free from stress.
- The column is perfectly elastic, homogeneous and isotropic.

4.7.1 Eulers Critical Load for Long Columns

Case (1) Both ends hinged

Consider a long column with both ends hinged subjected to critical load P as shown.



Consider a section at a distance x from the origin. Let y be the deflection of the column at this section. Bending moment in terms of load P and deflection y is given by

$$M = -P\,y \qquad \text{------------ (1)}$$

We can also write that for beams/columns the bending moment is proportional to the curvature of the beam, which, for small deflection can be expressed as

__ ___ or ___…………..(2)

where E is the Young's modulus and I is the moment of Inertia.

Substituting eq.(1) in eq.(2)

$$-P\,y = EI\,\frac{d^2y}{dx^2}$$

$$\text{or} \qquad \frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right)y = 0$$

This is a second order differential equation, which has a general solution form of

$$y = C_1 \sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cos\left(x\sqrt{\frac{P}{EI}}\right) \qquad \text{------(3)}$$

where C_1 and C_2 are constants. The values of constants can be obtained by applying the boundary conditions:

(i) y = 0 at x = 0. That is, the deflection of the column must be zero at each end since it is pinned at each end. Applying these conditions (putting these values into the eq. (3)) gives us the following results: For y to be zero at x =0, the value of C_2 must be zero (since cos (0) = 1).

(i) Substituting y = 0 at x = L in eq. (3) lead to the following.

$$0 = C_1 \sin\left(L\sqrt{\frac{P}{EI}}\right)$$

While for y to be zero at x = L, then either C_1 must be zero (which leaves us with no equation at all, if C_1 and C_2 are both zero), or

$$\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

which results in the fact that

$$\left(L\sqrt{\frac{P}{EI}}\right) = n\pi$$

or $$L\sqrt{\frac{P}{EI}} = n\pi \qquad \text{where } n = 0,1,2,2\ldots$$

or $$P = \frac{n^2\pi^2 EI}{L^2}$$

Taking least significant value of n, i.e. n= 1

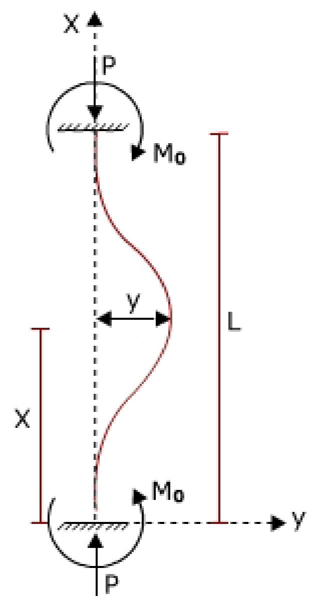
We have $$P = \frac{\pi^2 EI}{L^2}$$

or $$P_E = \frac{\pi^2 EI}{l_e^2}$$

where l_e =L.

Case (2) Both ends fixed

Consider a long column with both ends fixed subjected to critical load P as shown.



Consider a section at a distance x from the origin. Let y be the deflection of the column at this section. Bending moment in terms of load P, fixed end moment M 0 and deflection y is given by

$$M = -P\,y + M_0 \qquad \text{-------------(1)}$$

We can also write that for beams/columns the bending moment is proportional to the curvature of the beam, which, for small deflection can be expressed as

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

or $$M = EI\frac{d^2y}{dx^2} \qquad \text{-------------(2)}$$

where E is the Young's modulus and I is the moment of Inertia.

Substituting eq.(1) in eq.(2)

$$-P\,y + M_0 = E\,I\,\frac{d^2y}{dx^2}$$

or $$\frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right) y = \frac{M_0}{EI}$$

This is a second order differential equation, which has a general solution form of

$$y = C_1 \sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \quad \text{---------} \; (3)$$

where C_1 and C_2 are constants. The values of constants can be obtained by applying the boundary conditions:

(i) y = 0 at x = 0. That is, the deflection of the column must be zero at near end since it is fixed. Applying this condition (putting these values into the eq. (3)) gives us the following result:

$$C_2 = -\frac{M_0}{P}$$

ii) At X = 0 =0, that is, the slope of the column must be zero, since it is fixed.

$$\frac{dy}{dx} = C_1\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right) - C_2\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right) \quad \text{-------}(4)$$

Substituting the boundary condition in eq. (4)

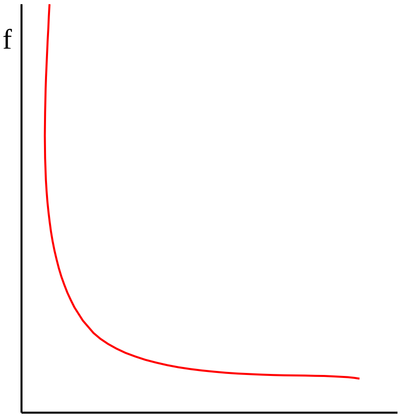
$$0 = C_1\sqrt{\frac{P}{EI}}$$

Hence, $$C_1 = 0$$

Substituting the constants C_1 and C_2 in eq. (3) leads to the following

$$y = -\frac{M_0}{P}\cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \qquad \text{---------(5)}$$

The variation of limiting stress 'f' versus slenderness ratio in the above equation is shown below.



The above plot shows that the limiting stress 'f' decreases as increases. In fact, when very small, limiting stress is is close to infinity, which is not rational. Limiting stress cannot be greater than the yield stress of the material.

1. Eulers formula determines the critical load, not the working load. Suitable factor of safety (which is about 1.7 to 2.5) should be considered to obtain the allowable load.

4.7.2 Rankine's critical Load

$$\frac{1}{P_R} = \frac{1}{P_C} + \frac{1}{P_E} \qquad \ldots\ldots\ldots\ldots \quad (1)$$

Where, P_R = Rankine's critical load

$P_C = f_C A$ = Crushing load for short columns

$P_E = \frac{\pi^2 E I}{l_e^2}$ = Euler's critical load for long columns

Rankine Gordon Load is given by the following empirical formula,

This relationship is assumed to be valid for short, medium and long columns. This relation can be used to find the load carrying capacity of a column subjected to crushing and/or buckling.

From eq. (1)

Substituting P_C and P_E in the above relation

$$P_R = \frac{f_c\ A}{1 + \left[\dfrac{\dfrac{f_c\ A}{\pi^2 E I}}{l_e^2}\right]} = \frac{f_c\ A}{1 + \left(\dfrac{f_c}{\pi^2 E}\right)\left[\dfrac{l_e^2\ A}{I}\right]}$$

Since $\dfrac{I_{min}}{A} = (r_{min})^2$

$$P_R = \frac{f_c\ A}{1 + a\left[\dfrac{l_e}{r_{min}}\right]^2}$$