Engineering Mathematics-IV

SUBJECT CODE: 15MAT41

Module –II: Numerical Methods and special functions

Dr Shivalingappa Sangappa Benchalli
Professor,
Department of Mathematics,
Basaveshwar Engineering college, Bagalkot-587102
Karnataka State, India.
Email: sbenchalli@gmail.com, (M): +918762644634,
Tel.: +918354-234060, Fax.: +918354-234204, 233905,
web.: http://www.becbgk.edu

Syllabus:

MODULE-II **Numerical Methods:** Numerical solution of second order ordinary differential equations, Runge-Kutta method and Milne's method. **Special Functions:** Series solution-Frobenious method. Series solution of Bessel's differential equation leading to $J_n(x)$-Bessel's function of first kind. Basic properties and orthogonality. Series solution of Legendre's differential equation leading to $P_n(x)$-Legendre polynomials. Rodrigue's formula, problems	L3	10

Text Books:

1. *B.S. Grewal: Higher Engineering Mathematics, Khanna Publishers, 43rd Ed., 2015.*
2. *E. Kreyszig: Advanced Engineering Mathematics, John Wiley & Sons, 10th Ed., 2015.*

Learning outcomes:

Upon successful completion of Numerical solution of second order ordinary differential equations, by Runge Kutta method and Milne's method, it is expected that a student will be able to do the following.

- Familiar with Fourth order Runge-Kutta method and Milne's method.
- Find the numerical solutions of second order ordinary differential equations, using Runge_kutta method and Milne's method.

Introduction:

Is it necessary to study Numerical Analysis: or
Why it is necessary to study Numerical analysis:

In the real world, no system behaves in a linear manner. There is no ideal material, ideal support condition and a perfect structure. Due to imperfections (defect) the behavior can always be described by a set of non-linear equations only. These equations cannot be solved analytically except in some trivial cases, and one has to resort to numerical analysis to find solutions. Nava day's it has become an important tool to solve a wide spectrum of nonlinear problems that arise in many practical situations.

What is Numerical Analysis?.

Numerical analysis is the development and study of procedures for solving problems with a computer.

Advantages:

1. A major advantage for numerical analysis is that a numerical answer can be obtained even when a problem has "no analytical" solution.
2. Numerical results can be plotted to show some of the behavior of the solution.
3. Another important distinction is that result from numerical analysis is an approximation, but results can be made as accurate as desired. (There are limitations to the achievable level of accuracy, because of the way that computers do arithmetic).

Second order differential equation:

Consider the second order differential equation $\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$ by writing dy/dx = z, it can be reduced to two first order simultaneous differential equations.

$$\frac{dy}{dx} = z, \qquad \frac{dz}{dx} = f(x, y, z)$$

These two equations can be solved using fourth order Runge-Kutta method.

Example1: Using Runge - Kutta method, solve $\frac{d^2y}{dx^2} = xy'^2 - y^2$ for x = 0.2 correct to 4 decimal places. Initial conditions are x = 0, y = 1, y' = 0.

Solution: Let $\frac{dy}{dx} = z = f(x, y, z)$, Then $\frac{dz}{dx} = xz^2 - y^2 = \phi(x, y, z)$

We have $x_0 = 0$, $y_0 = 1$, $z_0 = 0$, h = 0.2. Using k_1, k_2, k_3, k_4 for f(x, y, z) and l_1, l_2, l_3 for ϕ(x, y, z) Runge - Kutta formulae become

$$k_1 = hf(x_0, y_0, z_0) = 0.2(0) = 0 \qquad l_1 = h\phi(x_0, y_0, z_0) = 0.2(-1) = -0.2$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) = 0.2(-0.1) = -0.02 \ ,$$

$$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) = 0.2(-0.999) = -0.1998$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) = 0.2(-0.0999) = -0.02,$$

$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) = 0.2(-0.9791) = -0.1958$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2(-0.1958) = -0.0392.$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2(0.9527) = -0.1905$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 - 0.0199 = 0.9801$$

$$y' = z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 0 - 0.1970 = -0.1970$$

Example 2: Given $y'' + xy' + y = 0$ y(0) = 1 y'(0) = 0, obtain y for x = 0, 0.1, 0.3 by any method. Further, continue the solution by Milnes method to calculate y(0.4).

Solution: put y' = z , the given equation reduces to the simultaneous equations

$$z' + xz + y = 0, \; y' = z$$

We use Taylor's series method to find y, differentiating the given equation n times, we get

$$y_{n+2} + xy_{n+1} + ny_n + y_n = 0$$

at $x = 0$

$$(y_{n+2})_0 = -(n+1)(y_n)_0$$

$$\therefore y(0) = 1 \text{ gives } \quad y_2(0) = -1, y_4(0) = 32, y_6(0) = -15, \ldots.$$

$$\text{and } \; y_1(0) = 1 \text{ yields } \quad y_3(0) = 0, y_5(0) = 0, y_7(0) = 0, \ldots. = 0$$

Expanding y(x) by Taylor's series, we have

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \ldots.$$

$$y(x) = 1 - \frac{x^2}{2!} + 3\frac{x^4}{4!} - \frac{5x^3}{6!}x^6 + \ldots\ldots \qquad \text{--------(2)}$$

$$\text{and } z(x) = y'(x) = -x + \frac{1}{2}x^3 - \frac{1}{8}x^5 = \ldots\ldots = -xy \qquad \text{--------(3)}$$

From (2) y(0.1) = 0.995, y(0.2) = 0.9802, y(0.3) = 0.956

From (3) we have, z(0.1) = - 0.0995, z(0.2) = - 0.196, z(0.3) = - 0.2863

Also from (1) $z'(x) = -(xz + y)$

$z'(0.1) = -0.985, \; z'(0.2) = -0.941, z'(0.3) = -0.87.$

Applying Milne's predictor formula, first to z and then to y, we obtain

$$z(0.4) = z(0) + \frac{4}{3}(0.1)(2z'(0.1) - z'(0.2) + 2z'(0.3))$$

$$= 0 + \frac{0.4}{3}(-1.97 + 0.941 - 1.74) = -0.3692$$

$$y(0.4) = y(0) + \frac{4}{3}(0.1)(2y'(0.1) - y'(0.2) + 2y'(0.3))$$

$$= 0 + \frac{0.4}{3}(-0.199 + 0.196 - 0.5736) = 0.9231$$

Also $z'(0.4) = (x(0.4)z(0.4) + y(0.4)) = (0.4(-0.3692) + 0.9231) = -0.7754.$

Now Applying Milne's corrector formula, we get

$$z(0.4) = z(0.2) + \frac{h}{3}(z'(0.2) + 4z'(0.3) + z'(0.4))$$

$$= -0.196 + \frac{0.1}{3}(-0.941 - 3.48 - 0.7754) = -0.3692$$

$$y(0.4) = y(0.2) + \frac{h}{3}(y'(0.2)4y'(0.3) + y'(0.4))$$

$$= 0.9802 + \frac{0.1}{3}(-0.196 - 01.1452 - 0.3692) = 0.9232$$

Hence $y(0.4) = 0.9232$, and $z(0.4) = -0.3692$.

Additional Resources: Please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_4th_method.html

http://numericalmethods.eng.usf.edu

Special Functions:

Introduction:

We are familiar with the solution of linear differential equations with Constant coefficients. The solution involves elementary functions such as e^{ax}, sin(ax), cos(ax) etc. However, linear differential equations with variable coefficients, which arise from physical problems, do not permit such solutions. Such equations can be solved by numerical methods, But in many cases it is easier to find a solution in the form of an infinite converge series. The series solution of certain differential equations give rise to special functions such as Bessel's function, Legendre's polynomial, Hermite's polynomial, Chebyshev polynomial. These special functions have many applications in Engineering.

Series solution of differential equation:

To solve the equation of the form $p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0$

Where $p_0(x)$, $p_1(x)$, $p_2(x)$ are polynomials in x, in terms of infinite convergent series.

Validity of series solution:

Every differential equation of the form $p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0 ----(1)$

Does not have series solution. As such we find the conditions under which the above equation admits of the series solution.Dividing equation (1) by $p_0(x)$, we have

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

where $p(x) = \frac{p_1(x)}{p_0(x)}$ and $q(x) = \frac{p_2(x)}{p_0(x)}$

Ordinary point:

$x = 0$ is called an ordinary point of (1) if $p_0(x) \neq 0$, otherwise it is called a singular point.

When $x = 0$ is an ordinary point of (1) its every solution can be expressed as a series of the form

$$y = a_0 + a_1 x + a_2 x^2 + ------ = \sum_{k=0}^{\infty} a_k x^k$$

Singular point: When x = 0 is called a singular point of (1) if $p_0(0) = 0$. If $x\,p(x)$ and $x^2 q(x)$ possess derivatives of all orders in the neighborhood of x = 0 , then x = 0 is called a regular singular point of (1)

When $x = 0$ is a regular singular point of (1) at least one of its solution can be expressed as

$$y = x^m\left(a_0 + a_1 x + a_2 x^2 + -----\right) = \sum_{k=0}^{\infty} a_k x^{m+k}$$

Where m may be a positive or negative integer or a fraction.

When $x = 0$ is an irregular singular point of (1) , then the differential equation of (1) has no series solution of the form

$$y = x^m\left(a_0 + a_1 x + a_2 x^2 + -----\right) = \sum_{k=0}^{\infty} a_k x^{m+k}$$

Series solution When x = 0 is an ordinary point of the equation:

$$p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0 ------(1)$$

Let $$y = a_0 + a_1 x + a_2 x^2 + ------ = \sum_{k=0}^{\infty} a_k x^k -----(2)$$

be the solution of (1) then find dy/dx , d^2y / dx^2 Substitute the values of

$y, \frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in (1)

equate to zero the coefficients of various powers of x and find a_2, a_3,a_4 …… in terms of a_0 and a_1.Equate to zero the coefficient of x^n. The relation so obtained is called the recurrence relation. Give different values to n in the recurrence relation to determine various a_i's in terms of a_0 and a_1.

Substitute the values of a_2, a_3, a_4, …. In equation (2) to get the series solution of (1) having a_0 and a_1 as arbitrary constants.

Examples : To solve the equation $\frac{d^2y}{dx^2} + xy = 0$ -------- (1)

Solution: since x = 0 is an ordinary point of (1)

Let its series solution be $\mathbf{y = a_0 + a_1x + a_2x^2 + ------ = \sum_{k=0}^{\infty} a_k x^k}$ -----(2)

Then find first and second derivative of (2) we get

$$\frac{dy}{dx} = \mathbf{a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1}}$$

$$\frac{d^2y}{dx^2} = 2.1\mathbf{a_2} + 3\,2\mathbf{a_3x} + 4.3\mathbf{a_4x^2 .... + n(n-1)a_nx^{n-2}} + ...$$

Substituting these values in the given equation we get

$$2.1\mathbf{a_2} + 3\,2\mathbf{a_3x} + 4.3\mathbf{a_4x^2 .... + n(n-1)a_nx^{n-2} + ... + x(a_0 + a_1x + a_2x^2 + ... + a_nx^n)} = 0$$

$$2.1\mathbf{a_2} + (3\,2\mathbf{a_3 + a_0)x} + (4\,3\mathbf{a_4 + a_1)x^2} + (5\,4\mathbf{a_4 + a_1)x^3 + ... + [(n+2)(n+1)a_{n+2} + a_{n-1}]x^n} + = 0$$

Equating to zero the coefficients of the various powers of x we get

$$\mathbf{a_2} = 0$$

$$3\,2\mathbf{a_3} + \mathbf{a_0} = 0 \text{ i.e } \mathbf{a_3} = -\frac{\mathbf{a_0}}{3!}$$

$$4\,3\mathbf{a_4} + \mathbf{a_1} = 0, \text{ i.e } \mathbf{a_4} = -\frac{2\mathbf{a_1}}{4}$$

$$5\,4\mathbf{a_5} + \mathbf{a_2} = 0, \text{i.e } \mathbf{a_5} = -\frac{\mathbf{a_2}}{5\,4} \text{ and so on..}$$

$$\text{In general, } (n+2)(n+1)a_{n+2} + a_{n-1} = 0, \text{ i.e } a_{n+2} = -\frac{a_{n-1}}{(n+2)(n+1)}$$

which is the recurrence relation

put n = 4,5,6.....in the recurrence relation we get

$$a_6 = -\frac{a_3}{6\,5} = \frac{4\mathbf{a_0}}{6!}; \quad a_7 = -\frac{a_4}{7\,6} = \frac{5\,2\,\mathbf{a_1}}{7!}; \text{ and so on....}$$

substituting these values in (2) we get

$$y = a_0\left(1 - \frac{x^3}{3!} + \frac{1\,4x^6}{6!} - \frac{1\,4\,7\,x^9}{9!} ...\right) + a_1\left(x - \frac{2x^4}{4!} + \frac{2\,5\,x^7}{7!} + ---\right)$$

Example2: To solve the equation $\frac{d^2y}{dx^2} + x^2y = 0$ -------- (1)

Solution: since $x = 0$ is an ordinary point of (1)

Let its series solution be $$\mathbf{y = a_0 + a_1x + a_2x^2 + ------ = \sum_{k=0}^{\infty} a_k x^k} ------(2)$$

Then find the first and second derivatives of (2) we get

$$\mathbf{\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + ...+ na_nx^{n-1} = \sum_{k-1}^{\infty} ka_kx^{k-1}}$$

$$\mathbf{\frac{d^2y}{dx^2} = 2.1a_2 + 3\,2a_3x + 4.3a_4x^2 ....+ n(n-1)a_nx^{n-2} + ...= \sum_{k=2}^{\infty} k(k-1)a_kx^{k-2}}$$

Substituting the values of y and d^2y/dx^2 in the given differential equation (1) we get

$$\mathbf{\sum_{k=2}^{\infty} k(k-1)a_kx^{k-2} + x^2\sum_{k=0}^{\infty} a_kx^k = 0}$$

$$\mathbf{[2.1a_2 + 3\,2a_3x + 4.3a_4x^2 ....+ (n+2)(n+1)a_{n+2}x^n + ...]+}$$
$$\mathbf{+ x^2(a_0 + a_1x + a_2x^2 + ...+ a_{n-2}x^{n-2} + ...) = 0}$$
$$\mathbf{2.1a_2 + (3\,2a_3)x + (4\,3a_4 + a_0)x^2 + (5\,4a_4 + a_1)x^3 + ..}$$
$$\mathbf{.....+ [(n+2)(n+1)a_{n+2} + a_{n-2}]x^n + ... = 0}$$

Equating to zero the coefficients of the various powers of x we get

$\mathbf{a_2 = 0}$ $\mathbf{a_3 = 0}$ and so on..

In general, $(n+2)(n+1)a_{n+2} + a_{n-2} = 0,$

i.e $a_{n+2} = -\frac{a_{n-2}}{(n+2)(n+1)}$

which is the recurrence relation

put $n = 2,3,4,5,6.....$ in the recurrence relation we get

$$a_4 = -\frac{a_0}{3\,4} = \frac{\mathbf{4a_0}}{\mathbf{6!}}; \quad a_5 = -\frac{\mathbf{a_1}}{45}; \text{ and so on....}$$

substituting these values in (2) we get

$$y = a_0(1 - \frac{x^4}{3\,4} + \frac{x^8}{3\,4\,7\,8}...) + a_1(x - \frac{x^5}{4\;5} + \frac{x^9}{4\,5\,8\,9} + ---)$$

Examples3: To solve the equation $(1+\mathbf{x}^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$ -------- (1)

Solution: since x = 0 is an ordinary point of (1)

Let its series solution be $\mathbf{y = a_0 + a_1x + a_2x^2 + ------ = \sum_{k=0}^{\infty} a_k x^k}$ $-----(2)$

then $\frac{\mathbf{dy}}{\mathbf{dx}} = \mathbf{a}_1 + 2\mathbf{a}_2\mathbf{x} + 3\mathbf{a}_3\mathbf{x}^2 + \ldots\ldots\ldots + \mathbf{na_nx^{n-1}} = \sum_{k-1}^{\infty} \mathbf{ka_kx^{k-1}}$

$$\frac{\mathbf{d^2y}}{\mathbf{dx^2}} = 2.1\mathbf{a}_2 + 3\,2\mathbf{a}_3\mathbf{x} + 4.3\mathbf{a}_4\mathbf{x}^2 \ldots\ldots\ldots + \mathbf{n(n-1)a_nx^{n-2}} + \ldots\ldots\ldots = \sum_{k=2}^{\infty} \mathbf{k(k-1)a_kx^{k-2}}$$

Substituting the values of y, dy/dx and d^2y/dx^2 in the given differential equation (1) and equating the various powers of x obtain $a_2 = (a_0/2)$, a3 = 0

Equating to zero the coefficients of x^n , we get $a_{n+2} = -\frac{(n-1)}{(n+2)}\mathbf{a_n}$

Put n = 2,3,4,5…successively we get

$$a_4 = -\frac{(1)}{(4)}\mathbf{a}_2 = -\frac{\mathbf{a}_0}{8}; \; a_5 = -\frac{(2)}{(5)}\mathbf{a}_3 = 0;$$

$$a_6 = -\frac{(1)}{(2)}\mathbf{a}_4 = \frac{\mathbf{a}_0}{16}\ldots\ldots\ldots$$

substiting the values of a_i's in the assumed solution we get

$$y = a_0\left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^2}{128} + \ldots\ldots\right) \text{ required equation}$$

Probenius method:

Series solution when x = 0 is a regular singularity of the equation

$$\mathbf{p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0 ----------(1)}$$

Let $\mathbf{y = x^m(a_0 + a_1x + a_2x^2 + -----a_nx^n + \ldots\ldots) = \sum_{k=0}^{\infty} a_kx^{m+k}}$ $---(2)$

be the solution of (1) then

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \sum_{k=0}^{\infty}\mathbf{(m+k)a_kx^{m+k-1}}$$

$$\frac{\mathbf{d^2y}}{\mathbf{dx^2}} = \sum_{k=0}^{\infty}\mathbf{(m+k)(m+k-1)a_kx^{m+k-2}}$$

substiting the values of y, $\frac{dy}{dx}$ **and** $\frac{d^2y}{dx^2}$ **in** (1)

Equate to zero the coefficient of lowest power of x. This gives a quadratic equation in m which is known as the indicial equation. Equate to zero the coefficients of other powers of x to find $a_1, a_2, a_3 \ldots\ldots$ in terms of a_0. Substitute the values of a_1, a_2, $a_3 \ldots\ldots$ in (2) to get the series solution of (1), since the complete solution must have two independent arbitrary constants. The method of complete solution depends on the nature of roots of the indicial equation.

Case 1:

When the roots m_1 and m_2 of the indicial equation are distinct and not differing by an integer. The complete solution is

$$\mathbf{y = c_1 (y)_{m1} + c_2 (y)_{m2}}$$

Case 2: When the roots m_1 and m_2 of the indicial equation are equal the complete solution is $\mathbf{y = c_1 (y)_{m1} + c_2 (\partial y/ \partial m)_{m1}}$

Case 3: When the roots m_1 and m_2 ($m_1 < m_2$) of the indicial equation are distinct and differ by an integer. Let some of the coefficients of y series become infinite when $m = m_1$ (smaller of the two roots). Replace $\mathbf{a_0}$ by $\mathbf{b_0 (m - m_1)}$ in the series for y.

The complete solution is $\mathbf{y = c_1 (y)_{m1} + c_2 (\partial y/ \partial m)_{m1}}$

The solution corresponding to $\mathbf{m = m_2}$ (greater of the two roots) is a constant multiple of the solution corresponding to $\mathbf{m = m_1}$.

Example : solve in series the equation $2x^2 \frac{d^2y}{dx^2} + (2x^2 - x)\frac{dy}{dx} + y = 0$

Solution : since x = 0 is a regular singular point of the given equation. Let its series solution be

$$y = x^m\left(a_0 + a_1x + a_2x^2 + ------\right) = \sum_{k=0}^{\infty} a_k x^{m+k}$$

Then

$$\frac{dy}{dx} = \sum_{k=0}^{\infty}(m+k)a_k x^{m+k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty}(m+k)(m+k-1)a_k x^{m+k-2}$$

substiting the values of $y, \frac{dy}{dx}$ **and** $\frac{d^2y}{dx^2}$ **in** the given equation, we get

$$2x^2\sum_{k=0}^{\infty}(m+k)(m+k-1)a_k x^{m+k-2} + (2x^2 - x)\sum_{k=0}^{\infty}(m+k)a_k x^{m+k-1} + \sum_{k=0}^{\infty} a_k x^{m+k} = 0.$$

$$2x^2\sum_{k=0}^{\infty}(m+k)(m+k-1)a_k x^{m+k-2}+(2x^2-x)\sum_{k=0}^{\infty}(m+k)a_k x^{m+k-1}+\sum_{k=0}^{\infty}a_k x^{m+k}=0.$$

$$2x^2[m(m-1)a_0x^{m-2}+(m+1)(m)a_1x^{m-1}+(m+2)(m+1)a_2x^m+(m+3)(m+2)a_3x^{m-1}+\ldots]+$$

$$+(2x^2-x)[(m)a_0x^{m-1}+(m+1)a_1x^m+(m+2)a_2x^{m+1}+(m+3)a_3x^{m+2}+\ldots.]+$$

$$+a_0x^m+a_1x^{m+1}+a_2x^{m+2}+\ldots\ldots=0.$$

The lowest power of x is x^m. Equating to zero the coefficient of x^m, we get $2m(m-1)a_0 - ma_0 + a_0 = 0$. Or $(2m^2 -3m + 1)\, a_0 = 0$ or $(2m -1)(m -1) = 0$ since $a_0 \neq 0$. Which is the indicial equation. Its roots are m = ½ , 1.Equating to zero the coefficient of x^{m+1}, we get

$2m(m+1)a_1 + 2ma_0 - (m+1)a_1 + a_1 = 0$ or

$(2m^2 +m)\, a_1 + 2ma_0 = 0$ or $m\,[(2m+1)a_1 +2a_0] = 0$ or

$(2m+1)\, a_1 + 2a_0 = 0$ since $m \neq 0$ or

$a_1 = - (2 / (2m+1))\, a_0$

Equating to zero the coefficient of x^{m+2}

$2\,(m+2)(m+1)\, a_2 + 2\,(m+1)\, a_1 - (m+2)\, a_2 + a_2 = 0.$ or

$(2m^2 + 5m + 3)a_2 + 2\,(m + 1)a_1 = 0$ or

$(2m + 3)(m + 1)\, a_2 + 2\,(m + 1)\, a_1 = 0$

$(2m+3)(m+1)\, a_2 + 2\,(m+1)\, a_1 = 0$

$[(2m+3)\, a_2 + 2\, a_1]\,(m+1) = 0$ or

$(2m+3)\, a_2 + 2\, a_1 = 0$ since $(m+1) \neq 0$ or

$a_2 = - (2 / (2m +3))\, a_1 = - (4/ (2m+1)(2m+3))\, a_0$

Equating to zero the coefficient of x^{m+3} **,** we get

$2\,(m+3)(m+2)\, a_3 + 2\,(m+2)\, a_2 - (m+3)\, a_3 + a_3 = 0$ or

$(2m_2 + 9m + 10)\, a_3 + 2\,(m+2)\, a_2 = 0$ or

$(2m + 5)(m+2)\, a_3 + 2\,(m+2)\, a_2 = 0$ or

$(2m + 5)\, a_3 + 2\, a_2 = 0$ since $(m+2) \neq 0$

$a_3 = - (8 / (2m+1)(2m+3)(2m+5))\, a_0$ and so on….

When m = ½ : $a_1 = - a_0$, $a_2 = - (a_0/2)$, $a_3 =- (a_0/6)$ etcThe first solution is

$$\mathbf{y_1 = a_0 x^{1/2}\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} - - - - -\right)}$$

Therefore the second independent solution is

$$y_2 = a_0 x\left(1 - \frac{2}{3}x + \frac{2^2x^2}{35} - \frac{2^3x^3}{357} + - - - - - -\right)$$

Hence the complete solution is $y = c_1 y_1 + c_2 y_2$

Example1: Home work

Solve in series the equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

Hint: Since x = 0 is a regular singular point of the given equation. And roots are m = -2 and 2.

Example 2 : solve in series the equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$$

Hint: Roots are **m = 0, 0**

Example 3: Obtain the series solution of the equation $x(1-x)\frac{d^2y}{dx^2} - (1+3x)\frac{dy}{dx} - y = 0$

Hint: Roots are m = 0,2

Bessel's equation:

The second order differential equation given as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 - - - - - - - (1)$$

is known as Bessel's equation. Where the solution to Bessel's equation yields Bessel's functions of the first kind and second kind as follows $y = A\,J_n(x) + B\,Y_n(x)$ Where A and B are arbitrary constants.

Bessel function :

1. First kind: Jn(x) is the solution to Bessel's equation is referred to as a Bessel's function of the first kind.
2. Second kind: Yn(x) in the solution to Bessel's equation is referred to as a Bessel's function of the second kind or sometimes the Weber function or the Neumann function.

Equation (1) is often encountered when solving boundary value problem, such as separable solutions to Laplace's equation or the Helmholtz equation, especially when working in cylindrical or spherical coordinates. The constant n determines the order of the Bessel's function found in the solution to Bessel's differential equation and can take on any real number value. For cylindrical problems the order of the Bessel function is an integer value n while for spherical problems the order is of half integer value n+1/2. Since Bessel's differential equation is a second order equation there must be 2 linearly independent solutions.

Example:

Solve $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0-------(1)$$

Since x = 0 is a regular singular point of the given equation, Let its series solution be

$$y=x^m\left(a_0+a_1x+a_2x^2+-------\right)=\sum_{k=0}^{\infty}a_kx^{m+k}$$

Differentiate above equation w.r.t. x two times then substitute these values in the equation (1).

$$\frac{dy}{dx}=\sum_{k=0}^{\infty}(m+k)a_kx^{m+k-1}$$

$$\frac{d^2y}{dx^2}=\sum_{k=0}^{\infty}(m+k)(m+k-1)a_kx^{m+k-2}$$

substiting the values of $y,\frac{dy}{dx}$ **and** $\frac{d^2y}{dx^2}$ **in** the given equation, we get

$$x^2\sum_{k=0}^{\infty}(m+k)(m+k-1)a_kx^{m+k-2}+(x)\sum_{k=0}^{\infty}(m+k)a_kx^{m+k-1}+(x^2-n^2)\sum_{k=0}^{\infty}a_kx^{m+k}=0.$$

$$x^2\sum_{k=0}^{\infty}(m+k)(m+k-1)a_kx^{m+k-2}+(x)\sum_{k=0}^{\infty}(m+k)a_kx^{m+k-1}+(x^2-n^2)\sum_{k=0}^{\infty}a_kx^{m+k}=0.$$

$$\sum_{k=0}^{\infty}(m+k)(m+k-1)a_kx^{m+k}+\sum_{k=0}^{\infty}(m+k)a_kx^{m+k}+x^2\sum_{k=0}^{\infty}a_kx^{m+k}-n^2\sum_{k=0}^{\infty}a_kx^{m+k}=0.$$

$$\sum_{k=0}^{\infty}(m+k)(m+k)a_kx^{m+k}-\sum_{k=0}^{\infty}(m+k)a_kx^{m+k}+\sum_{k=0}^{\infty}(m+k)a_kx^{m+k}+-n^2\sum_{k=0}^{\infty}a_kx^{m+k}+\sum_{k=0}^{\infty}a_kx^{m+k+2}=0.$$

$$\sum_{k=0}^{\infty}\left[(m+k)^2-(m+k)+(m+k)-n^2\right]a_kx^{m+k}+\sum_{k=0}^{\infty}a_kx^{m+k+2}=0.$$

$$\sum_{k=0}^{\infty}\left[(m+k)^2-n^2\right]a_kx^{m+k}+\sum_{k=0}^{\infty}a_kx^{m+k+2}=0.$$

The lowest power of x is x^m corresponding to k = 0. Equating to zero, the coefficient of x^m, we get indicial equation

$m^2-n^2=0$, (since $a_0\neq 0$) implies $m=\pm n$

Equating to zero the coefficient of next term i.e. x^{m+1} we get

$[(m+1)^2-n^2]a_1=0$ implies $a_1=0$, since $[(m+1)^2-n^2]a_1\neq 0$ for $m=\pm n$.

Equating to zero the coefficient of x m+k+2 we get the recurrence relation

$[(m+k+2)^2-n^2]a_{K+2}+a_k=0$ or

$$a_{k+2} = \frac{a_k}{(m-n+k+2)(m+n+k+2)}$$

$$a_{k+2} = \frac{a_k}{(m-n+k+2)(m+n+k+2)}$$

put $k = 1,3,5,\ldots\ldots$ we get $a_3 = a_5 = a_7 = \ldots. = 0.$

put $k = 0, 2, 4,\ldots\ldots$ we get

$$a_2 = \frac{a_0}{(m-n+2)(m+n+2)}$$

$$a_4 = \frac{a_2}{(m-n+4)(m+n+4)} =$$

$$= \frac{a_0}{(m-n+4)(m+n+4)(m-n+2)(m+n+2)}$$

and so on ......,

substituting all the values values in the assumed solution, we get

$$\therefore y = a_0 x^m \left[1 - \frac{x^2}{(m+2)^2 - n^2} + \frac{x^4}{[(m+2)^2 - n^2][(m+4)^2 - n^2]} \ldots\ldots\ldots\ldots\right] - - - - - (a)$$

Depending upon the values of n , we get different types of solutions

Case I : When n ≠0 or n ≠ an integer . In this case , we get two independent solutions for m = n and m = - n.

For m = n we get

$$\therefore y_1 = a_0 x^n \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4^2 2!\,[(n+1)(n+2)]} - \frac{x^6}{4^3 3!\,[(n+1)(n+2)(n+3)]} \ldots\ldots\ldots\ldots\right] - -(2)$$

For m = -n we get

$$\therefore y_2 = a_0 x^{-n} \left[1 - \frac{x^2}{4(-n+1)} + \frac{x^4}{4^2 2!\,[(-n+1)(-n+2)]} - \frac{x^6}{4^3 3!\,[(-n+1)(-n+2)(-n+3)]} \ldots\ldots\ldots\ldots\right] - -(3)$$

If we take $a_0 = \dfrac{1}{2^n \Gamma(n+1)}$

Then the solution given by (2) is called the Bessel function of the first kind of order n and is denoted by Jn(x). Thus (WKT $n\Gamma n = \Gamma(n+1)$)

$$\therefore J_n(x) = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \frac{1}{1!\,\Gamma(n+2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(n+3)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma(n+4)}\left(\frac{x}{2}\right)^6 \ldots\ldots\right], (n > 0)$$

$$\text{i.e. } J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)} - - - - - (4)$$

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{r!\,\Gamma(-n+r+1)} - - - - - (5)$$

Which is called the Bessel's function of the first kind of order –n. Hence complete solution of the Bessel's equation (1) may be expressed in the form y = A J_n (x) + B J_{-n} (x). Where A and B are arbitrary constants.

Case 2: When n = 0, the Bessel's equation takes the form

$$\therefore \mathbf{y} = \mathbf{x}^{\mathbf{m}}\left[1 - \frac{\mathbf{x}^2}{(\mathbf{m}+2)^2} + \frac{\mathbf{x}^4}{[(\mathbf{m}+2)^2][(\mathbf{m}+4)^2]} - \frac{\mathbf{x}^6}{[(\mathbf{m}+2)^2][(\mathbf{m}+4)^2][(\mathbf{m}+6)^2]} + \ldots\ldots\ldots\ldots\right]$$

If m = 0 the first solution is given by

$$\mathbf{J}_0(\mathbf{x}) = \sum_{\mathbf{k}=0}^{\infty} (-1)^{\mathbf{k}} \left(\frac{\mathbf{x}}{2}\right)^{2\mathbf{k}} \frac{1}{(\mathbf{k}!)^2} \quad ,$$

in equation (4) put n = 0 , r = k, $\Gamma(\mathbf{k}+1) = \mathbf{k}!$

Which is Bessel function of the kind of order zero.

We know that

$$\mathbf{y} = \mathbf{a}_0\mathbf{x}^{\mathbf{m}}\left[1 - \frac{\mathbf{x}^2}{(\mathbf{m}+2)^2 - \mathbf{n}^2} + \frac{\mathbf{x}^4}{[(\mathbf{m}+2)^2 - \mathbf{n}^2][(\mathbf{m}+4)^2 - \mathbf{n}^2]}\ldots\ldots\ldots\ldots\right] - - - - - (\mathbf{a})$$

Now differentiate above equation with respect to m , we get

$$\therefore \frac{\partial \mathbf{y}}{\partial \mathbf{m}} = \mathbf{x}^{\mathbf{m}} \log \mathbf{x}\left[1 - \frac{\mathbf{x}^2}{(\mathbf{m}+2)^2} + \frac{\mathbf{x}^4}{[(\mathbf{m}+2)^2][(\mathbf{m}+4)^2]}\ldots\ldots\ldots\ldots\right] +$$

$$+ \mathbf{x}^{\mathbf{m}}\left[\frac{\mathbf{x}^2}{(\mathbf{m}+2)^2}\frac{2}{(\mathbf{m}+2)} - \frac{\mathbf{x}^4}{[(\mathbf{m}+2)^2][(\mathbf{m}+4)^2]}\left\{\frac{2}{(\mathbf{m}+2)} + \frac{2}{(\mathbf{m}+4)}\right\} + \ldots.\right]$$

The second independent solution is given by $\left(\frac{\partial \mathbf{y}}{\partial \mathbf{m}}\right)_{\mathbf{m}=0}$.

The complete solution of the Bessel equation of order zero is y = AJ_0(x) + B y_0 (x) , where y_0(x) is called Bessel function of the second kind of order zero or Neumann function.

Case 3 : When n is an integer , the two functions J_n (x) J_{-n}(x) are not independent but are connected by the relation

$$\mathbf{J}_{-\mathbf{n}}(\mathbf{x}) = (-1)^{\mathbf{n}} \mathbf{J}_{\mathbf{n}}(\mathbf{x})$$

$$\mathbf{w.k.t.}\ \mathbf{J}_{-\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{k}=0}^{\infty} (-1)^{\mathbf{k}} \left(\frac{\mathbf{x}}{2}\right)^{-\mathbf{n}+2\mathbf{r}} \frac{1}{\mathbf{k}!\,\Gamma(-\mathbf{n}+\mathbf{k}+1)}$$

Since Γ(a negative integer or zero) tends to ∞, each term in the summation is zero as long as $-n+k+1 \le 0$ i.e $k \le n-1$. and $\Gamma(-n+k+1)$ is finite when $k \ge n$.

Put k = n+r , we observe that when k = n , r = 0, k tends to ∞, r tends ∞ .

$$\mathbf{J_{-n}(x)} = \sum_{\mathbf{r}=0}^{\infty} \frac{(-1)^{\mathbf{n+r}}}{\mathbf{(n+r)!\Gamma(r+1)}} \left(\frac{\mathbf{x}}{2}\right)^{\mathbf{n+2r}}$$

$$= \mathbf{(-1)^n} \sum_{\mathbf{r}=0}^{\infty} \frac{(-1)^{\mathbf{r}}}{\mathbf{(n+r)!\Gamma(r+1)}} \left(\frac{\mathbf{x}}{2}\right)^{\mathbf{n+2r}}$$

since r and n are integer $\Gamma(r+1) = r!$ and $(n+r)! = \Gamma(n+r+1)$

$$\mathbf{J_{-n}(x)} = \mathbf{(-1)^n} \sum_{\mathbf{r}=0}^{\infty} \frac{(-1)^{\mathbf{r}}}{\mathbf{(r)!\Gamma(n+r+1)}} \left(\frac{\mathbf{x}}{2}\right)^{\mathbf{n+2r}}$$

$$= \mathbf{(-1)^n J_n(x).}$$

Now , when n is an integer , y_2 fails to give a solution for positive values of n and y_1 fails to give a solution for negative values of n.Let us find an independent solution of Bessel's equation (1) when n is an integer. Let y = u(x) J_n (x) be a solution of equation (1) , n is integer. Then differentiate y two times we get

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \mathbf{u'\,J_n(x)} + \mathbf{u}\,J_n'$$

$$\frac{\mathbf{d^2y}}{\mathbf{dx^2}} = \mathbf{u''\,J_n(x)} + \mathbf{u'}\,J_n' + \mathbf{u'}\,J_n' + \mathbf{u}\,J_n''$$

$$= \mathbf{u''\,J_n(x)} + 2\mathbf{u'}\,J_n' + \mathbf{u}\,J_n''$$

Substituting the values of y , dy/dx , d^2y/dx^2 in equation (1) we get

$$\mathbf{x}^2\left(\mathbf{u''\,J_n(x)} + 2\mathbf{u'}\,J_n' + \mathbf{u}\,J_n''\right) + \mathbf{x}\left(\mathbf{u'\,J_n} + \mathbf{u}J_n'\right) + \mathbf{(x^2 - n^2)uJ_n} = 0$$

or

$$\mathbf{u}\left[\mathbf{x}^2 J_n'' + \mathbf{x}J_n' + \mathbf{(x^2 - n^2)J_n}\right] + \mathbf{x^2u''\,J_n(x)} + 2\mathbf{x^2u'}\,J_n' + \mathbf{xu'\,J_n} = 0$$

or

$$\mathbf{x^2u''\,J_n(x)} + 2\mathbf{x^2u'}\,J_n' + \mathbf{xu'\,J_n} = 0 \quad \because J_n \text{ is a solution of equation (1).}$$

$$\mathbf{x^2u''\,J_n(x)} + 2\mathbf{x^2u'}\,J_n' + \mathbf{xu'\,J_n} = 0$$

dividing throughout by $\mathbf{x^2u'\,J_n}$ **we get**

$$\frac{\mathbf{u''}}{\mathbf{u'}} + 2\frac{J_n'}{\mathbf{J_n}} + \frac{1}{\mathbf{x}} = 0$$

$$\textbf{i.e. } \frac{d}{dx}\left(\mathbf{\log u'}\right) + 2\frac{d}{dx}\left(\mathbf{\log J_n}\right) + \frac{d}{dx}\left(\mathbf{\log x}\right) = 0$$

$$\textbf{or } \frac{d}{dx}\left(\mathbf{\log u'} + 2\,\mathbf{\log J_n} + \mathbf{\log x}\right) = 0$$

$$\textbf{or } \frac{d}{dx}\left(\mathbf{\log}\left(\mathbf{u'\,J_n^2x}\right)\right) = 0$$

Integrating w.r.t. x we get

$$\mathbf{\log}\left(\mathbf{u'\,J_n^2x}\right) = \mathbf{\log B} \quad \therefore \mathbf{u'} = \frac{\mathbf{B}}{\mathbf{J_n^2x}} \text{ or } u = B\int \frac{\mathbf{dx}}{\mathbf{J_n^2x}} + \mathbf{A}$$

Substituting the value of u in the assumed solution $y = u(x)\, J_n(x)$

$$y = \left[B\int \frac{dx}{J_n^2 x} + A \right] J_n(x)$$

$$\textbf{or} \quad y = A\,J_n(x) + B y_n(x), \quad \text{where } y_n(x) = J_n(x) \int \frac{dx}{J_n^2 x}$$

The function $y_n(x)$ is called Bessel function of the second kind of order n or Neumann function.

Find the value of $J_{1/2}(x)$:

$$\text{w.K.T.} \quad J_n(x) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2K}$$

$$\textbf{put } n = \frac{1}{2} \text{ we get and w.k.t } \Gamma n = (n-1)\Gamma(n-1)$$

$$J_{1/2}(x) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!\,\Gamma(k+3/2)} \left(\frac{x}{2}\right)^{1/2+2K}$$

$$= \frac{1}{\Gamma(3/2)}\left(\frac{x}{2}\right)^{1/2} - \frac{1}{\Gamma(5/2)}\left(\frac{x}{2}\right)^{5/2} + \frac{1}{2!\,\Gamma(7/2)}\left(\frac{x}{2}\right)^{7/2} + \ldots\ldots$$

$$= \left(\frac{x}{2}\right)^{1/2} \left[\frac{1}{(1/2)\Gamma(1/2)} - \frac{1}{(3/2)(1/2)\Gamma(1/2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2(5/2)(3/2)(1/2)\Gamma(1/2)}\left(\frac{x}{2}\right)^4 + \ldots \right]$$

$$J_{1/2}(x) = \frac{\sqrt{x}}{\sqrt{2}\,\Gamma(1/2)} \left[\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!} - \ldots \right]$$

$$= \sqrt{\frac{x}{2\pi}} \; \frac{2}{x} \left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots \right]$$

$$= \sqrt{\frac{2}{\pi x}}(\sin x).$$

$$\therefore J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}(\sin x).$$

Find the value of $J_{-1/2}(x)$:

$$\text{w.K.T.} \quad J_{-n}(x) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!\,\Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{-n+2K}$$

$$\textbf{put } n = \frac{1}{2} \text{ we get and w.k.t } \Gamma n = (n-1)\Gamma(n-1)$$

$$J_{-1/2}(x) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!\,\Gamma(k+1/2)} \left(\frac{x}{2}\right)^{-1/2+2K}$$

$$= \frac{1}{\Gamma(1/2)}\left(\frac{x}{2}\right)^{-1/2} - \frac{1}{\Gamma(3/2)}\left(\frac{x}{2}\right)^{3/2} + \frac{1}{2!\,\Gamma(5/2)}\left(\frac{x}{2}\right)^{7/2} + \ldots\ldots$$

$$= \left(\frac{x}{2}\right)^{-1/2} \left[\frac{1}{\Gamma(1/2)} - \frac{1}{(1/2)\Gamma(1/2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2(3/2)(1/2)\Gamma(1/2)}\left(\frac{x}{2}\right)^4 + \ldots \right]$$

$$J_{-1/2}(x) = \frac{\sqrt{2}}{\sqrt{x}\Gamma(1/2)}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ..\right]$$

$$= \sqrt{\frac{2}{x\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ......\right]$$

$$= \sqrt{\frac{2}{\pi x}}(\cos x).$$

$$\therefore J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}(\cos x).$$

Recurrence relation of Bessel functions:

The following recurrence relations connect Bessel functions of different orders and are very helpful in the solution of problems involving Bessel functions.

1.Prove that : $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

Proof : W.K.T. $J_n(x) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2K}$

$$x^n J_n(x) = \sum_{k=0}^{n-1} \frac{(-1)^k x^{2n+2k}}{2^{n+2k} k!\,\Gamma(n+k+1)}$$

differentiate both sides we get

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{k=0}^{n-1} \frac{(-1)^k (2n+2k) x^{2n+2k-1}}{2^{n+2k} k!\,\Gamma(n+k+1)}$$

$$= x^n \sum_{k=0}^{n-1} \frac{(-1)^k (n+k) x^{n+2k-1}}{2^{n+2k-1} k!\,(n+k)\Gamma(n+k)}$$

$$= x^n \sum_{k=0}^{n-1} \frac{(-1)^k}{2^{n+2k-1} k!\,\Gamma(n-1+k+1)}\left(\frac{x}{2}\right)^{n+2k-1}$$

$$= x^n J_{n-1}(x)$$

$$\therefore \frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

2.Prove that : $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

Proof : W.K.T. $J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2K}$

$$x^{-n} J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{n+2k} k!\,\Gamma(n+k+1)}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{k=1}^{\infty} \frac{(-1)^k (2k) x^{2k-1}}{2^{n+2k} k(k-1)!\,\Gamma(n+k+1)}$$

$$= -x^{-n} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{n+2k-1}}{2^{n+2k-1} (k-1)!\,\Gamma(n+k+1)}$$

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}\sum_{k=1}^{\infty}\frac{(-1)^r x^{n+2r-1}}{2^{n+2r+1} r!\,\Gamma(n+r+2)} \quad \text{where } r = k-1$$

$$= -x^{-n}\sum_{r=0}^{\infty}\frac{(-1)^r x}{2^{n+2r+1} r!\,\Gamma(n+1+r+1)}\left(\frac{x}{2}\right)^{n+2r+1}$$

$$= -x^{-n}J_{n+1}(x)$$

$$\therefore \frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x).$$

In particular, when $x = 0$ we have

$$\frac{d}{dx}\left[J_0(x)\right] = -J_1(x) \quad \text{or} \quad J_0' = -J_1$$

3. **Prove** that: $2nJ_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$

$$J_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2K}$$

$$2nJ_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k\, 2n+2k-2k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2K}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k\, 2(n+k)}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2K} - \sum_{k=0}^{\infty}\frac{(-1)^k\, 2k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2K}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k\, 2(n+k)}{k!\,(n+k)\Gamma(n+k)}\left(\frac{x}{2}\right)^{n+2K} - \sum_{k=0}^{\infty}\frac{(-1)^k\, 2k}{k(k-1)!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2K}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k x\left(\frac{x}{2}\right)^{n+2K-1}}{k!\,\Gamma(n+k)} + \sum_{k=0}^{\infty}\frac{(-1)^{k-1} x\left(\frac{x}{2}\right)^{n+2K-1}}{(k-1)!\,\Gamma(n+k+1)}$$

$$= x\left[\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{n-1+2K}}{k!\,\Gamma(n-1+k+1)} + \sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{x}{2}\right)^{n+1+2r}}{r!\,\Gamma(n+1+r+1)}\right] \quad \text{where } r = k-1$$

$$= x\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

$$\therefore 2nJ_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

4. Prove that: $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) + J_{n+1}(x)\right]$ (similar to previous proof)

Example1: Prove that $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$.

Proof: From the recurrence relation (1) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) - - - - - (1)$

$$\left[x^n J_n'(x) + nx^{n-1}J_n(x)\right] = x^n J_{n-1}(x)$$

Dividing by x^{n-1} we get $xJ'(x) = xJ_{n-1}(x) - nJ_n(x) - - - - - (3)$ Which is the required equation

Example2: Prove that $xJ'(x) = nJ_n(x) - xJ_{n+1}(x)$ --(1)

Proof: From recurrence relation (2) we get $\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$ ------(2)

$$\left[x^{-n}J_n'(x) - nx^{-n-1}J_n(x)\right] = -x^{-n}J_{n+1}(x)$$

Multiplying by x^{n+1} we get

$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$ ------(4) Which is the required equation.

Example3: Prove that $4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+1}(x)$

Proof: W.K.T from recurrence relation (4) $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ ----(1)

Differentiate (1), we get $2J_n''(x) = J_{n-1}'(x) - J_{n+1}'(x)$ ----(2)

Changing n to (n-1) and (n+1) in equation (1) we get $2J_{n-1}'(x) = J_{n-2}(x) - J_n(x)$

$$2J_{n+1}'(x) = J_n(x) - J_{n+2}(x)$$

Subtracting ,we get $2\left[J_{n-1}'(x) - J_{n+1}'(x)\right] = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$

Using equation (2) , the above equation can be written as

$$4J_n'' = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$$

Example 4: show that $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right)$

Solution: From recurrence relation (3) $2nJ_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$

It can be written as $J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$

Put $n = 3/2$ in the above equation we get

$$J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x)$$

$$\therefore J_{5/2}(x) = \frac{3}{x}\left[\frac{1}{x}J_{1/2}(x) - J_{-1/2}(x)\right] - J_{1/2}(x)$$

$$= \left(\frac{3}{x^2} - 1\right)J_{1/2}(x) - \frac{3}{x}J_{-1/2}(x)$$

$$= \left(\frac{3-x^2}{x^2}\right)\sqrt{\frac{2}{\pi x}}\sin x - \frac{3}{x}\sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3-x^2}{x^2}\right)\sin x - \frac{3}{x}\cos x\right]$$

Example5: Show that $J_4(x)=\left(\frac{48}{x^3}-\frac{8}{x}\right)J_1(x)+\left(1-\frac{24}{x^2}\right)\frac{3}{x}J_0(x)$

Solution: From recurrence relation (3) $2nJ_n(x)=x[J_{n-1}(x)+J_{n+1}(x)]$

It can be written as $J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x)---(1)$

Put $n=1$ in the above equation we get

$$J_2(x)=\frac{2}{x}J_1(x)-J_0(x)-----(2)$$

Put $n=2$ in eqn(1) we get

$$J_3(x)=\frac{4}{x}J_2(x)-J_1(x)-----(3)$$

put $n=3: J_4(x)=\frac{6}{x}J_3(x)-J_2(x)$

$$=\frac{6}{x}\left[\frac{4}{x}J_2(x)-J_1(x)\right]-J_2(x) \quad \text{using (3)}$$

$$=\left(\frac{24}{x^2-1}\right)J_2(x)-\frac{6}{x}J_1(x)$$

$$=\left(\frac{24}{x^2-1}\right)J_2(x)-\frac{6}{x}J_1(x)$$

$$=\left(\frac{24}{x^2-1}\right)\left[\frac{2}{x}J_1(x)-J_0(x)\right]-\frac{6}{x}J_1(x)$$

$$=\left(\frac{48}{x^3}-\frac{2}{x}-\frac{6}{x}\right)J_1(x)+\left(1-\frac{24}{x^2}\right)J_0(x)$$

$$=\left(\frac{48}{x^3}-\frac{8}{x}\right)J_1(x)+\left(1-\frac{24}{x^2}\right)J_0(x)$$

Example6: Prove that $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)]=x[J_n^2(x)-J_{n+1}^2(x)]$

Proof.:

$$\text{LHS}=\frac{d}{dx}[xJ_n(x)J_{n+1}(x)]=\frac{d}{dx}[x^{-n}J_n(x)x^{n+1}J_{n+1}(x)]$$

$$=x^{-n}J_n(x)\frac{d}{dx}[x^{n+1}J_{n+1}(x)]+x^{n+1}J_{n+1}(x)\frac{d}{dx}[x^{-n}J_n(x)]--(1)$$

Now $\frac{d}{dx}[x^nJ_n(x)]=x^nJ_{n-1}(x)$

changing n to $(n+1)$, we have $\frac{d}{dx}[x^{n+1}J_{n+1}(x)]=x^{n+1}J_n(x)$

Also w.k.t. $\frac{d}{dx}[x^{-n}J_n(x)]=-x^{-n}J_{n+1}(x)$

From equation (1) we have $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)]=$

$$=x^{-n}J_n(x)[x^{n+1}J_n(x)]+x^{n+1}J_{n+1}(x)[-x^{-n}J_{n+1}(x)]$$

$$=x[J_n^2(x)-J_{n+1}^2(x)].$$

Example7: **Prove that** $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=2\left[\frac{n}{x}J_n^2(x)-\frac{n+1}{x}J_{n+1}^2(x)\right]$

Proof : LHS $=\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=2J_n(x)J_n'(x)+2J_{n+1}(x)J_{n+1}'(x)--(1)$

From example 1 we have (using example 2) $J_n'(x)=-\frac{n}{x}J_n(x)+J_{n-1}(x)-----(2)$

Replace n by (n+1) in equation (2) we get $J_{n+1}'(x)=-\frac{n+1}{x}J_{n+1}(x)+J_n(x)-----(4)$

Substituting the values of $J_n'(x)\ J_{n+1}'(x)$ from equation (2) and (4) in equation (1)

$$\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=2J_n(x)\left[\frac{n}{x}J_n(x)-J_{n+1}(x)\right]+2J_{n+1}(x)\left[-\frac{n+1}{x}J_{n+1}(x)+J_n(x)\right]$$

$$=2\left[\frac{n}{x}J^2{}_n(x)-\frac{n+1}{x}J^2{}_{n+1}(x)\right].$$

Generating Function for $J_n(x)$:

We shall now show that Bessel functions of various orders can be derived as co-efficient of various powers of t in the expansion of the function $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$

i.e. to prove that $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}t^nJ_n(x).$

$$\textbf{W.K.T. } e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+.......+\frac{x^n}{n!}+,....$$

$$e^{\frac{x}{2}(t-\frac{1}{t})}=e^{\frac{xt}{2}}e^{\frac{-x}{2t}}$$

$$=\left[1+\left(\frac{xt}{2}\right)+\frac{1}{2!}\left(\frac{xt}{2}\right)^2+\frac{1}{3!}\left(\frac{xt}{2}\right)^3+...+\frac{x^nt^n}{2^nn!}+\frac{x^{n+1}t^{n+1}}{2^{n+1}(n+1)!}+\frac{x^{n+2}t^{n+2}}{2^{n+2}(n+2)!}...\right]\text{x}$$

$$\text{x}\left[1-\left(\frac{x}{2t}\right)+\frac{1}{2!}\left(\frac{x}{2t}\right)^2-\frac{1}{3!}\left(\frac{x}{2t}\right)^3+...+(-1)^n\frac{x^n}{2^nt^nn!}+\frac{(-1)^{n+1}x^{n+1}}{2^{n+1}t^{n+1}(n+1)!}+\frac{(-1)^{n+2}x^{n+2}}{2^{n+2}t^{n+2}(n+2)!}...\right]$$

The coefficient of t^n in the product

$$e^{\frac{x}{2}(t-\frac{1}{t})}=e^{\frac{xt}{2}}e^{\frac{-x}{2t}}=\frac{1}{n!}\left(\frac{x}{2}\right)^n-\frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2}+\frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4}-...=J_n(x)$$

Similarly the coefficient of t^{-n} in the product

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{xt}{2}} e^{\frac{-x}{2t}} = \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^{n+2}}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} \ldots$$

$$= (-1)^n\left[\frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} \ldots\right]$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

$$= (-1)^n J_n(x) = J_{-n}(x)$$

Thus we have proved that $J_n(x)$ and $J_{-n}(x)$ are respectively the coefficients of t^n and t^{-n} in the expansion of the function $e^{\frac{x}{2}(t-\frac{1}{t})}$

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(x) + tJ_1(x) + t^2J_2(x) + \ldots + t^nJ_n(x) + \ldots$$

$$\ldots\ldots\ldots\ldots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + \ldots + t^{-n}J_{-n}(x) =$$

$$= \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

This shows that Bessel functions of various orders can be derived as coefficient of different powers of t in the expansion of $e^{\frac{x}{2}(t-\frac{1}{t})}$

For this reason, it is known as the generating function of Bessel function.

Integral form of Bessel Function:

Prove that $J_n(x) = \frac{1}{\pi}\int_0^{\pi}[\cos(n\theta - x\sin\theta)]\,d\theta,$ for all integral values of n

w.k.t. $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

$$= J_0(x) + tJ_1(x) + t^2J_2(x) + \ldots + t^nJ_n(x) + \ldots$$

$$\ldots\ldots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + \ldots + t^{-n}J_{-n}(x) + \ldots.$$

$$= J_0(x) + (t - \frac{1}{t})J_1(x) + (t^2 + \frac{1}{t^2})J_2(x) + (t^3 - \frac{1}{t^3})J_3(x) + \ldots\ldots(1)$$

Put t = cos θ + i sin θ , By De Movier's theorem

$$t^n = \cos n\theta + i\sin n\theta \quad \text{and} \quad \frac{1}{t^n} = \cos n\theta - i\sin n\theta$$

$$t^n + \frac{1}{t^n} = 2\cos n\theta \quad \text{and} \quad t^n - \frac{1}{t^n} = 2i\sin n\theta$$

Substituting these values in (1), we have

$$e^{i x \sin\theta} = J_0(x) + 2 i \sin\theta J_1(x) + 2\cos 2\theta J_2(x) + 3 i \sin 3\theta J_3(x) \ldots + \ldots\ldots (2)$$

$$\text{since } e^{i x \sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta)$$

Equating the real and imaginary parts in (2), we get

$$\cos(x\sin\theta) = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \ldots\ldots] --- (3)$$

$$\sin(x\sin\theta) = 2[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \ldots\ldots] --- (4)$$

These are known as Jacobi series.

Multiplying both sides of (3) by connθ and integrating w.r.t θ between. The limits 0 to π (when n is odd, all terms on the RHS vanish; when n is even, all terms on the RHS except the one containing $\cos n\theta$ vanish).

we get $$\int_0^{\pi} \cos(x\sin\theta)\cos n\theta\, d\theta = \begin{cases} 0 & , \text{ when n is odd} \\ \pi J_n(x) & , \text{ when n is even} \end{cases} ----(5)$$

Similarly, multiplying (4), by sinn θ and integrating w.r.t. θ between the Limits 0 to π we get

$$\int_0^{\pi} \sin(x\sin\theta)\sin n\theta\, d\theta = \begin{cases} \pi J_n(x), & \text{when n is odd} \\ 0 & , \text{ when n is even} \end{cases} ----(6)$$

Adding (5) and (6) we get

$$\int_0^{\pi} \left[\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta\right] d\theta = \pi J_n(x)$$

$$\therefore J_n(x) = \frac{1}{\pi}\int_0^{\pi} \left[\cos(n\theta - x\sin\theta)\right] d\theta, \text{ for all integral values of n}$$

Orthogonality of Bessel functions :

we shall prove that $$\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = \begin{cases} 0 & , \text{ when } \alpha \neq \beta \\ \frac{1}{2}[J_{n+1}(\alpha)]^2 & , \text{ when } \alpha = \beta \end{cases}$$

where α, β are the roots of $J_n(x) = 0$.

We know that the solution of the equation

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 ------(1)$$

$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 ------(2)$$

are $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ respectively, Multiplyi ng equation (1) by $\frac{v}{x}$ and (2) by $\frac{u}{x}$

we obtain $$xvu'' + vu' + \alpha^2 uvx - \frac{n^2uv}{x} = 0 \text{ and}$$

$$xuv'' + uv' + \beta^2 uvx - \frac{n^2uv}{x} = 0$$

On subtracting we obtain $\mathbf{x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)uvx} = 0$

$$\textbf{i.e.}\ \frac{\mathbf{d}}{\mathbf{dx}}\{\mathbf{x(u'v - uv')}\} = (\beta^2 - \alpha^2)\mathbf{uvx}$$

Now integrating both sides w.r.t x between o and 1

$$[\mathbf{x(u'v - uv')}]_0^1 = (\beta^2 - \alpha^2)\int_0^1 \mathbf{uvxdx}$$

$$\textbf{i.e.}(\mathbf{u'v - uv'})_{\mathbf{x}=1} - 0 = (\beta^2 - \alpha^2)\int_0^1 \mathbf{uvxdx} - - - (3)$$

$$\text{since } u = J_n(\alpha x)$$

$$\therefore u' = \frac{d}{dx}[J_n(\alpha x)] = \frac{d}{d(\alpha x)}[J_n(\alpha x)]\frac{d(\alpha x)}{\mathbf{dx}} = \alpha J_n'(\alpha x)$$

Similarly $v = J_n(\beta x)$

$$\therefore v' = \frac{d}{dx}[J_n(\beta x)] = \frac{d}{d(\beta x)}[J_n(\beta x)]\frac{d(\beta x)}{\mathbf{dx}} = \beta J_n'(\beta x)$$

Substituting these values in (3), we get

$$\alpha J_n'(\alpha)J_n(\beta) - \beta J_n(\alpha)J_n'(\beta) = (\beta^2 - \alpha^2)\int_0^1 \mathbf{x}J_n(\alpha x)J_n(\beta x)\mathbf{dx}$$

$$\textbf{i.e.}\int_0^1 \mathbf{x}J_n(\alpha x)J_n(\beta x)\mathbf{dx} = \frac{\alpha J_n'(\alpha)J_n(\beta) - \beta J_n(\alpha)J_n'(\beta)}{\beta^2 - \alpha^2} - -(4)$$

If α and β are distinct roots of $J_n(x) = 0$ then $J_n(\alpha) = J_n(\beta) = 0$

and (4) reduces to $\int_0^1 \mathbf{x}J_n(\alpha x)J_n(\beta x)\mathbf{dx} = 0 - - - (5)$

This is known as the orthogonality relation of Bessel functions when $\beta = \alpha$, the right side of equation (4) is of (0 / 0) form. Its value can β be found by considering α as a root of $J_n(x) = 0$ and β as a variable Approaching α, then equation (4) gives

$$\lim_{\beta\to\alpha}\int_0^1 \mathbf{x}J_n(\alpha x)J_n(\beta x)\mathbf{dx} = \lim_{\beta\to\alpha}\frac{\alpha J_n'(\alpha)J_n(\beta)}{\beta^2 - \alpha^2}$$

By L'Hospitals rule the numerator and denominator are differentiated Separately w.r.t. β. Thus we have

$$\int_0^1 \mathbf{x}[J_n^2(\alpha x)]\mathbf{dx} = \lim_{\beta\to\alpha}\frac{\alpha J_n'(\alpha)J_n'(\beta)}{2\beta} = \frac{1}{2}[J_n'(\alpha)]^2 - - - (6)$$

From the recurrence relation

$$J_n'(\mathbf{x}) = \frac{\mathbf{n}}{\mathbf{x}}J_n(\mathbf{x}) - J_{n+1}(\mathbf{x})$$

$$\therefore J_n'(\alpha) = \frac{\mathbf{n}}{\alpha}J_n(\alpha) - J_{n+1}(\alpha)$$

$$\Rightarrow J_n'(\alpha) = J_{n+1}(\alpha), \text{since } J_n(\alpha) = 0$$

Equation (6) becomes

$$\int_0^1 \mathbf{x}\left[J_n^2(\alpha x)\right]\mathbf{dx} = \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2$$

Solution of Legendre differential Equation:

Consider $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \text{ ---(1)}$

The coefficient of $\frac{d^2y}{dx^2}$ i.e $(1-x^2) \neq 0$ at $x = 0$

Assume series solution in the form $y = \sum_{r=0}^{\infty} a_r x^r$, ---(2)

$\therefore \frac{dy}{dx} = \sum_{r=1}^{\infty} a_r r\, x^{r-1}$, $\frac{d^2y}{dx^2} = \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}$

Equation (1) becomes

$$(1-x^2)\sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - 2x\sum_{r=1}^{\infty} a_r r\, x^{r-1} + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

$$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=2}^{\infty} a_r r(r-1)x^r - \sum_{r=1}^{\infty} 2a_r r\, x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

The first two terms of the above vanishes when r = 0, 1 and third term vanishes when r = 0 , we can write the above equation in the form

$$\sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=2}^{\infty} a_r r(r-1)x^r - 2\sum_{r=1}^{\infty} a_r r\, x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

Equate the coefficient of x^r to zero i.e in the first term replace r by r + 2 we get

$a_{r+2}(r+2)(r+1) - a_r(r)(r-1) - 2a_r(r) + n(n+1)a_r = 0$

i.e $a_{r+2}(r+2)(r+1) = a_r[r(r-1) + 2r - n(n+1)]$

$$a_{r+2} = \frac{\left[r^2 - r + 2r - n(n+1)\right]}{(r+2)(r+1)}a_r$$

$$a_{r+2} = -\frac{\left[n(n+1) - r^2 - r\right]}{(r+2)(r+1)}a_r \text{ ----(3)}$$

put r = 0 ,1,2,3...nin equation (3)

$$a_2 = \frac{-n(n+1)}{2}a_0,$$

$$a_3 = \frac{-(n^2+n-2)}{6}a_1 = \frac{-(n-1)(n+2)}{6}a_1$$

$$a_4 = \frac{-(n^2+n-6)}{12}a_2 = \frac{-(n-2)(n+3)}{12}\, x\, \frac{-n(n+1)}{2}a_0$$

$$= \frac{n(n+1)(n-2)(n+3)}{24}a_0$$

$$a_5 = \frac{-(n^2+n-12)}{20}a_3 = \frac{-(n-3)(n+4)}{20}\, x\, \frac{-(n-1)(n+2)}{6}a_1$$

$$= \frac{(n-1)(n+2)(n-3)(n+4)}{120}a_1 \text{ etc.}$$

Obviously $a_0, a_1 \neq 0$, otherwise we get a trivial solution $y = 0$. Substitute above values in the assumed solution (2).
i.e $y = a_0 + a_1x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots\ldots.$
i.e $y = (\mathbf{a_0} + \mathbf{a_2 x^2} + \mathbf{a_4 x^4} + \ldots.) + (\mathbf{a_1 x} + \mathbf{a_3 x^3} + \mathbf{a_5 x^5} \ldots..)$
Because a_2, a_4, --- are in terms of $\mathbf{a_0}$ and $a_3, a_5 \ldots$ are in terms of $\mathbf{a_1}$, we obtain

$$y = a_0\left[1 - \frac{\mathbf{n(n+1)}}{\mathbf{2!}}\mathbf{x}^2 + \frac{\mathbf{n(n+1)(n-2)(n+3)}}{\mathbf{4!}}\mathbf{x}^4 - \ldots\ldots.\right] +$$
$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}\mathbf{x}^3 + \frac{\mathbf{(n-1)(n+2)(n-3)(n+4)}}{\mathbf{5!}}\mathbf{x}^5 - \ldots.\right] - -(4)$$

Let $y_1(x)$, $y_2(x)$ respectively represent the two infinite series in (4)

i.e $y = a_0\, y_1(x) + a_1\, y_2(x)$ is the series solution of the Legendre's differential equation.

Legendre Polynomials:

If n is a positive even integer, $a_0\, y_1(x)$ reduces to a polynomial of degree n.
If n is a positive odd integer $a_1\, y_2(x)$ reduces to a polynomial of degree n
Otherwise these will give infinite series called Legendre functions of second kind

Take $a_0 = (-1)^{n/2} \cdot \frac{\mathbf{1.3.5\ldots.(n-1)}}{\mathbf{2.4.6\ldots n}}$ and $a_1 = (-1)^{n-1/2} \frac{\mathbf{1.3.5\ldots n}}{\mathbf{2.4.6\ldots(n-1)}}$

equation (4) becomes

$$p_n(x) = (-1)^{n/2} \cdot \frac{\mathbf{1.3.5\ldots.(n-1)}}{\mathbf{2.4.6\ldots n}}\left[1 - \frac{\mathbf{n(n+1)}}{\mathbf{2!}}\mathbf{x}^2 + \frac{\mathbf{n(n+1)(n-2)(n+3)}}{\mathbf{4!}}\mathbf{x}^4 - \ldots\ldots.\right]$$

when n is even

$$p_n(x) = (-1)^{n-1/2} \frac{\mathbf{1.3.5\ldots n}}{\mathbf{2.4.6\ldots(n-1)}}\left[x - \frac{(n-1)(n+2)}{3!}\mathbf{x}^3 + \frac{\mathbf{(n-1)(n+2)(n-3)(n+4)}}{\mathbf{5!}}\mathbf{x}^5 - \ldots.\right]$$

when n is odd.

Particular cases : put n = 0,1,2,3,4,5......

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = \frac{1}{2}(3\mathbf{x}^2 - 1), \quad P_3(x) = \frac{1}{2}(5\mathbf{x}^3 - 3\mathbf{x}),$$

$$\mathbf{p_4(x)} = \frac{1}{8}(35\mathbf{x}^4 - 30\mathbf{x}^2 + 3),$$

$$\mathbf{p_5(x)} = \frac{1}{8}(63\mathbf{x}^5 - 70\mathbf{x}^3 + 15\mathbf{x})$$

Example: Express $\mathbf{f(x) = x^4 + 3x^3 - x^2 + 5x - 2}$ in terms of Legendre polynomials

Solution :w.k.t. $\mathbf{p_4(x)} = \frac{1}{8}(35\mathbf{x}^4 - 30\mathbf{x}^2 + 3) = \frac{35}{8}\left[\mathbf{x}^4 - \frac{8}{35}\frac{15}{4}\mathbf{x}^2 + \frac{8}{35}\frac{3}{8}\right] = \frac{35}{8}\left[\mathbf{x}^4 - \frac{6}{7}\mathbf{x}^2 + \frac{3}{35}\right]$

$$\therefore \mathbf{x}^4 = \frac{8}{35}\mathbf{p_4(x)} + \frac{6}{7}\mathbf{x}^2 + \frac{3}{35}$$

$$P_3(x)=\frac{1}{2}\left(5x^3-3x\right)=\frac{5}{2}\left[x^3-\frac{3}{5}x\right]$$

$$\therefore x^3=\frac{2}{5}P_3(x)+\frac{3}{5}x, \text{ similarly } x^2=\frac{2}{3}P_2(x)+\frac{1}{3}.$$

$$\therefore f(x)=\left[\frac{8}{35}p_4(x)+\frac{6}{7}x^2-\frac{3}{35}\right]+3x^3-x^2+5x-2$$

$$=\frac{8}{35}p_4(x)+3x^3-\frac{1}{7}x^2+5x-\frac{73}{35}$$

$$=\frac{8}{35}p_4(x)+3\left[\frac{2}{5}p_3(x)+\frac{3}{5}x\right]-\frac{1}{7}\left[\frac{2}{3}P_2(x)+\frac{1}{3}\right]+5x-\frac{73}{35}$$

$$=\frac{8}{35}p_4(x)+\frac{6}{5}p_3(x)-\frac{2}{21}P_2(x)+\frac{34}{5}x-\frac{224}{105}$$

$$=\frac{8}{35}p_4(x)+\frac{6}{5}p_3(x)-\frac{2}{21}P_2(x)+\frac{34}{5}P_1(x)-\frac{224}{105}P_0(x), \because x=P_1(x),\ 1=P_0(x)$$

Generating Function for $P_n(x)$:

In the expansion $(1-2xz+z^2)^{-1/2}$ in powers of z, it can be shown the coefficient of Z^n is $P_n(x)$.

Hence $(1-2xz+z^2)^{-1/2}$ is called generating function for $p_n(x)$.

i.e. $(1-2xz+z^2)^{-1/2}=\sum_{n=0}^{\infty}P_n(x).z^n$

Proof : w.k.t. $(1-t)^{-n}=1+\frac{n}{1!}t+\frac{n(n+1)}{2!}t^2+\frac{n(n+1)(n+2)}{3!}t^3+....$

put $n=1/2$ in the above formula

$$(1-t)^{-1/2}=1+\frac{1}{2}t+\frac{\frac{1}{2}\left(\frac{3}{2}\right)}{2!}t^2+\frac{\frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{3!}t^3+....$$

$$=1+\frac{1}{2}t+\frac{1\,3}{2\,4}t^2+\frac{1\,3\,5}{2\,4\,6}t^3+\frac{1\,3\,5\,7}{2\,4\,6\,8}t^4+....+\frac{1\,3\,5\,7....(2n-1)}{2\,4\,6\,8......2n}t^n+...$$

i.e. $(1-2xz+z^2)^{-1/2}=\left[1-z(2x-z)\right]^{-1/2}$

w.k.t.

$$(1-t)^{-1/2}=1+\frac{1}{2}t+\frac{1\,3}{2\,4}t^2+\frac{1\,3\,5}{2\,4\,6}t^3+\frac{1\,3\,5\,7}{2\,4\,6\,8}t^4+....+\frac{1\,3\,5\,7....(2n-1)}{2\,4\,6\,8......2n}t^n+...$$

$$=1+\frac{1}{2}z(2x-z)+\frac{1\,3}{2\,4}z^2(2x-z)^2+\frac{1\,3\,5}{2\,4\,6}z^3(2x-z)^3+\frac{1\,3\,5\,7}{2\,4\,6\,8}z^4(2x-z)^4+$$

$$+\frac{1\,3\,5\,7\,9}{2\,4\,6\,8\,10}z^5(2x-z)^5++.........+\frac{1\,3\,5\,7\,9..(2n-5)}{2\,4\,6\,8\,10...(2n-4)}z^{n-2}(2x-z)^{n-2}+$$

$$+\frac{1\,3\,5\,7\,9..(2n-3)}{2\,4\,6\,8\,10...(2n-2)}z^{n-1}(2x-z)^{n-1}+\frac{1\,3\,5\,7\,9..(2n-1)}{2\,4\,6\,8\,10...(2n)}z^n(2x-z)^n+..$$

Expansion of the n^{th} term using the following formula

$$\text{w.k.t.}(x - a)^{\mathbf{n}} = \sum_{\mathbf{r}=0}^{\infty} (-1)^{\mathbf{r}}\ {}^{\mathbf{n}}\mathbf{c}_{\mathbf{r}}\ \mathbf{x}^{\mathbf{n-r}}\ \mathbf{a}^{\mathbf{r}} \text{ weget}$$

$$\therefore \frac{1\,3\,5\,7\,9..(2n-1)}{2\,4\,6\,8\,10...(2n)}\mathbf{z}^{\mathbf{n}}(2\mathbf{x}-\mathbf{z})^{\mathbf{n}} = \frac{1\,3\,5\,7\,9..(2n-1)}{2\,4\,6\,8\,10...(2n)}\mathbf{z}^{\mathbf{n}}\left\{\sum_{\mathbf{r}=0}^{\infty}(-1)^{\mathbf{r}}\ {}^{\mathbf{n}}\mathbf{c}_{\mathbf{r}}\ (2\mathbf{x})^{\mathbf{n-r}}\ \mathbf{z}^{\mathbf{r}}\right\}$$

$$= \frac{1\,3\,5\,7\,9..(2n-1)}{2\,4\,6\,8\,10...(2n)}\mathbf{z}^{\mathbf{n}}\left\{{}^{\mathbf{n}}\mathbf{c}_0\ (2\mathbf{x})^{\mathbf{n}} - {}^{\mathbf{n}}\mathbf{c}_1\ (2\mathbf{x})^{\mathbf{n}-1}\mathbf{z} + {}^{\mathbf{n}}\mathbf{c}_2\ (2\mathbf{x})^{\mathbf{n}-2}\mathbf{z}^2 + \mathbf{....}\right\}$$

Collect the coefficient of $\mathbf{z}^{\mathbf{n}}$ from the above equation, we get

$$\frac{1\,3\,5\,7\,9..(2n-1)}{2\,4\,6\,8\,10...(2n)}{}^{\mathbf{n}}\mathbf{c}_0\ (2\mathbf{x})^{\mathbf{n}} = \frac{1\,3\,5\,7\,9..(2n-1)}{2\,4\,6\,8\,10...(2n)}\{1\,2^n x^n\} =$$

$$= \frac{1\,3\,5\,7\,9..(2n-1)}{2^n\ n!}2^n x^n = \frac{1\,3\,5\,7\,9..(2n-1)}{n!}x^n$$

Expansion of the $(n-1)^{th}$ term using the following formula

$$(x - a)^{\mathbf{n}} = \sum_{\mathbf{r}=0}^{\infty} (-1)^{\mathbf{r}}\ {}^{\mathbf{n}}\mathbf{c}_{\mathbf{r}}\ \mathbf{x}^{\mathbf{n-r}}\ \mathbf{a}^{\mathbf{r}} \text{ is}$$

$$\therefore \frac{1\,3\,5\,7\,9..(2n-3)}{2\,4\,6\,8\,10...(2n-2)}\mathbf{z}^{\mathbf{n}-1}(2\mathbf{x}-\mathbf{z})^{\mathbf{n}-1} = \frac{1\,3\,5\,7\,9..(2n-3)}{2\,4\,6\,8\,10...(2n-2)}\mathbf{z}^{\mathbf{n}-1}\left\{\sum_{\mathbf{r}=0}^{\infty}(-1)^{\mathbf{r}}\ {}^{\mathbf{n}-1}\mathbf{c}_{\mathbf{r}}\ (2\mathbf{x})^{\mathbf{n}-1-\mathbf{r}}\ \mathbf{z}^{\mathbf{r}}\right\} =$$

$$= \frac{1\,3\,5\,7\,9..(2n-3)}{2\,4\,6\,8\,10...(2n-2)}\mathbf{z}^{\mathbf{n}-1}\left\{{}^{\mathbf{n}-1}\mathbf{c}_0\ (2\mathbf{x})^{\mathbf{n}-1} - {}^{\mathbf{n}-1}\mathbf{c}_1\ (2\mathbf{x})^{\mathbf{n}-2}\mathbf{z} + {}^{\mathbf{n}-1}\mathbf{c}_2\ (2\mathbf{x})^{\mathbf{n}-3}\mathbf{z}^2 + \mathbf{....}\right\}$$

Collect the coefficient of z^n from the above equation, we get

$$\frac{1\,3\,5\,7\,9..(2n-3)}{2\,4\,6\,8\,10...(2n-2)}\left\{-{}^{\mathbf{n}-1}\mathbf{c}_1\ (2\mathbf{x})^{\mathbf{n}-2}\right\} = -\frac{1\,3\,5\,7\,9..(2n-3)}{2\,4\,6\,8\,10...(2n-2)}\left\{(n-1)2^{n-2}x^{n-2}\right\} =$$

$$= -\frac{1\,3\,5\,7\,9..(2n-3)}{2^{n-1}\,(\mathbf{n}-1)!}\left\{(n-1)2^{n-2}x^{n-2}\right\}$$

multiplying both N^r and D^r by n (2n - 1) weget

$$= -\frac{1\,3\,5\,7\,9..(2n-3)\ n\ (2n-1)}{2^{n-1}\ n\,(\mathbf{n}-1)!\ (2n-1)}\left\{(n-1)2^{n-2}x^{n-2}\right\}$$

$$= -\frac{1\,3\,5\,7\,9..(2n-3)\ n\ (2n-1)}{2^{n-1}\ n!\ (2n-1)}\left\{(n-1)\ 2^{n-1}\ 2^{-1}\ x^{n-2}\right\}$$

$$= -\frac{1\,3\,5\,7\,9..(2n-3)\ n\ (2n-1)}{n!\ (2n-1)}\left\{(n-1)\ 2^{-1}\ x^{n-2}\right\}$$

$$= -\frac{1\,3\,5\,7\,9..(2n-3)\ (2n-1)}{n!}\ \frac{n(n-1)}{2(2n-1)}\left\{x^{n-2}\right\}$$

Similarly the coefficient of z^n in $\dfrac{1\,3\,5\,7\,9..(2n-5)}{2\,4\,6\,8\,10...(2n-4)}\mathbf{z}^{\mathbf{n}-2}(2\mathbf{x}-\mathbf{z})^{\mathbf{n}-2}$

Is

$$\frac{1\,3\,5\,7\,9..(2n-1)}{n!}\,\frac{\mathbf{n(n-1)(n-2)(n-3)}}{2\,4\,(2n-1)(2n-3)}x^{n-4}$$

$\therefore$ The coefficient of z^n in $(1-2xz+z^2)^{-1/2}$ is given by

$$\frac{1\,3\,5\,7\,9..(2n-1)}{n!}\left[\mathbf{x}^n-\frac{\mathbf{n(n-1)}}{2(2\mathbf{n}-1)}\mathbf{x}^{n-2}+\frac{\mathbf{n(n-1)(n-2)(n-3)}}{2\,4\,(2\mathbf{n}-1)(2\mathbf{n}-3)}\mathbf{x}^{n-4}+\ldots\right]=\mathbf{P_n(x)}$$

Thus in the expansion of $(1-2xz+z^2)^{-1/2}$, $p_1(x),\ p_2(x),\ p_3(x)\ldots P_n(x)$ are the coefficients of $z^1, z^2, z^3 \ldots z^n$ respectively.

$$\therefore (1-2xz+z^2)^{-1/2} = 1+ p_1(x)z+ p_2(x)z^2 + p_3(x)z^3+\ldots+P_n(x)\,z^n+\ldots..$$

$$=\sum_{n=0}^{\infty} P_n(x)\,z^n.$$

Example 1: Show that $P_n(1) = 1$.

$$\text{w.k.t.} \sum_{n=0}^{\infty} z^n p_n(x) = (1-2xz+z^2)^{-1/2} ----(1)$$

put $x = 1$ in equation (1) we get

$$\sum_{n=0}^{\infty} z^n p_n(1) = (1-2z+z^2)^{-1/2} = \left\{(1-z)^2\right\}^{-1/2} = (1-z)^{-1} = 1+\mathbf{z}+\mathbf{z}^2+\mathbf{z}^3+\ldots+\mathbf{z^n}+\ldots\ldots=\sum_{n=0}^{\infty} z^n$$

equating the coefficien ts of z^n we get $P_n(1)=1.$

Example 2 : Show that $P_n(-x) = (-1)^n P_n(x)$.

$$\text{w.k.t.} \sum_{n=0}^{\infty} z^n p_n(x) = (1-2xz+z^2)^{-1/2} ----(1)$$ replace x by (-x) in equation (1) we get

$$\sum_{n=0}^{\infty} z^n p_n(-x) = (1+2xz+z^2)^{-1/2} --(2)$$ again replace z by -z in (1)

$$\sum_{n=0}^{\infty} (-z)^n p_n(x) = (1+2xz+z^2)^{-1/2} \ \textbf{or}\ \sum_{n=0}^{\infty} (-1)^n z^n p_n(x) = (1+2xz+z^2)^{-1/2} --(3)$$

Fron (2) and (3)

$$\sum_{n=0}^{\infty} z^n p_n(-x) = \sum_{n=0}^{\infty} (-1)^n z^n p_n(x)$$ Equating the coefficient of z^n we have

$$p_n(-x) = (-1)^n p_n(x)$$

Rodrigue's formula : Show that $\mathbf{P_n(x)} = \dfrac{1}{2^n n!}\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{x}^2-1)^{\mathbf{n}}$

Let $u = (x^2 -1)$, First find the n^{th} derivative of u i.e. u_n is a solution of the Legendre's differential equation $(1-\mathbf{x}^2)\mathbf{y}''-2\mathbf{x}\mathbf{y}'+\mathbf{n(n+1)y}=0---(1)$

Differentiating **u** w.r.t. x

$$\frac{\mathbf{du}}{\mathbf{dx}} = \mathbf{u}_1 = \mathbf{n}(\mathbf{x}^2-1)^{\mathbf{n}-1}\,2\mathbf{x}$$

$$= \mathbf{n}(\mathbf{x}^2-1)^{\mathbf{n}}(\mathbf{x}^2-1)^{-1}\,2\mathbf{x}$$

$$= \frac{\mathbf{n}(\mathbf{x}^2-1)^{\mathbf{n}}\,2\mathbf{x}}{(\mathbf{x}^2-1)}$$

$$\mathbf{u}_1\left(\mathbf{x}^2-1\right)=\mathbf{n}\left(\mathbf{x}^2-1\right)^{\mathbf{n}}2\mathbf{x}$$

$= 2\text{nxu}$, differentiating w.r.t.x again

$$\left(\mathbf{x}^2-1\right)\mathbf{u}_2+2\mathbf{x}\mathbf{u}_1=2\mathbf{n}\left(\mathbf{x}\mathbf{u}_1+\mathbf{u}\right)$$

Differentiating n times above equation using Leibnitz's rule

$$(\text{uv})_{\mathbf{n}}=\text{u v}_n+\text{n u}_1\text{v}_{n-1}+\frac{\mathbf{n(n-1)}}{2!}\text{u}_2\text{v}_{n-2}+\ldots\ldots+\mathbf{u}_{\mathbf{n}}\mathbf{v}.$$

$$\therefore \frac{\mathbf{d}^{\mathbf{n}}}{\mathbf{dx}^{\mathbf{n}}}\left[\left(\mathbf{x}^2-1\right)\mathbf{u}_2\right]+2\frac{\mathbf{d}^{\mathbf{n}}}{\mathbf{dx}^{\mathbf{n}}}\left(\mathbf{x}\mathbf{u}_1\right)=2\mathbf{n}\frac{\mathbf{d}^{\mathbf{n}}}{\mathbf{dx}^{\mathbf{n}}}\left(\mathbf{x}\mathbf{u}_1\right)+2\mathbf{n}\mathbf{u}_{\mathbf{n}}$$

$$\mathbf{i.e}\left[\left(\mathbf{x}^2-1\right)\mathbf{u}_{\mathbf{n}+2}+\mathbf{n}\,2\text{x}\,\mathbf{u}_{\mathbf{n}+1}+\frac{\mathbf{n(n-1)}}{2}2\,\text{u}_n\right]+2\left[\text{x}\,\mathbf{u}_{\mathbf{n}+1}+\mathbf{n}\,1\,\text{u}_n\right]=$$

$$=2\mathbf{n}\left[\text{x}\,\mathbf{u}_{\mathbf{n}+1}+\mathbf{n}\,1\,\text{u}_n\right]+2\,\mathbf{n}\,\mathbf{u}_{\mathbf{n}}$$

$$\mathbf{i.e}\ \left(\mathbf{x}^2-1\right)\mathbf{u}_{\mathbf{n}+2}+2\mathbf{n}\,\text{x}\,\mathbf{u}_{\mathbf{n}+1}+\left(\mathbf{n}^2-\mathbf{n}\right)\text{u}_n+2\mathbf{x}\mathbf{u}_{\mathbf{n}+1}+2\mathbf{n}\text{u}_n=$$

$$=2\,\text{n x}\,\mathbf{u}_{\mathbf{n}+1}+2\mathbf{n}^2\,\text{u}_n+2\,\mathbf{n}\,\mathbf{u}_{\mathbf{n}}$$

$$\mathbf{i.e.}\left(\mathbf{x}^2-1\right)\mathbf{u}_{\mathbf{n}+2}+2\mathbf{x}\mathbf{u}_{\mathbf{n}+1}-\mathbf{n}^2\,\text{u}_n-\mathbf{n}\,\mathbf{u}_{\mathbf{n}}=0$$

$$\mathbf{i.e.}\left(\mathbf{x}^2-1\right)\mathbf{u}_{\mathbf{n}+2}+2\mathbf{x}\mathbf{u}_{\mathbf{n}+1}-\mathbf{n}\mathbf{u}_{\mathbf{n}}(\mathbf{n}+1)=0$$

or $\left(1-\mathbf{x}^2\right)\mathbf{u}_{\mathbf{n}+2}-2\mathbf{x}\mathbf{u}_{\mathbf{n}+1}+\mathbf{n(n+1)}\mathbf{u}_{\mathbf{n}}=0$**,** This can also be written as

$$\left(1-\mathbf{x}^2\right)\mathbf{u}_{\mathbf{n}}''-2\mathbf{x}\mathbf{u}_{\mathbf{n}}'+\mathbf{n(n+1)}\mathbf{u}_{\mathbf{n}}=0---(2)$$

$P_n(x)$ satisfies Legebdre’s differential equation is also a polinomial of degree n and hence un must be same as $P_n(x)$ but for some constant factor k.

$$\mathbf{i.e.}\ \text{P}_n(\text{x})=\text{k u}_n=\mathbf{k}\frac{\mathbf{d}^{\mathbf{n}}}{\mathbf{dx}^{\mathbf{n}}}\left(\mathbf{x}^2-1\right)^{\mathbf{n}}$$

$$\mathbf{i.e.}\ \text{P}_n(\text{x})=\text{k}\frac{\mathbf{d}^{\mathbf{n}}}{\mathbf{dx}^{\mathbf{n}}}\left[(\mathbf{x}-1)^{\mathbf{n}}(\mathbf{x}+1)^{\mathbf{n}}\right]\text{Apply Leib nitz's rule for RHS}$$

$$=\mathbf{k}\left[\begin{array}{l}(\mathbf{x}-1)^{\mathbf{n}}\left\{(\mathbf{x}+1)^{\mathbf{n}}\right\}_{\mathbf{n}}+\mathbf{n}\,\text{n}(\mathbf{x}-1)^{\mathbf{n}-1}\left\{(\mathbf{x}+1)^{\mathbf{n}}\right\}_{\mathbf{n}-1}+\\+\frac{\text{n(n-1)}}{2!}\mathbf{n(n-1)}(\mathbf{x}-1)^{\mathbf{n}-2}\left\{(\mathbf{x}+1)^{\mathbf{n}}\right\}_{\mathbf{n}-2}+\ldots+\left\{(\mathbf{x}+1)^{\mathbf{n}}\right\}_{\mathbf{n}}(\mathbf{x}+1)^{\mathbf{n}}\end{array}\right]--(3)$$

if $\text{z}=(\mathbf{x}-1)^{\mathbf{n}},\ \text{z}_1=\mathbf{n}(\mathbf{x}-1)^{\mathbf{n}-1},\ \text{z}_2=\mathbf{n(n-1)}(\mathbf{x}-1)^{\mathbf{n}-2}\ldots\ldots$

$\ldots z_{\mathbf{n}}=\mathbf{n(n-1)(n-2)}\ldots 2\,1\,(\mathbf{x}-1)^{\mathbf{n}-\mathbf{n}}$

$=\text{n!}\,(\mathbf{x}-1)^{0}=\mathbf{n}\,!$

$\therefore \left\{(\mathbf{x}-1)^{\mathbf{n}}\right\}_{\mathbf{n}}=\mathbf{n!}$

put $x = 1$ in (3) and all the terms in RHS becomes zero

except the reducing to $n!(1+1)^n=n!\,2^n$

i.e $P_n(1)=k\,n!\,2^n$ but $p_n(1)=1$

$1 \quad = k\,n!\,2^n \quad \therefore k=\dfrac{1}{n!\,2^n}$

since $P_n(x)=k\,u_n$

$$\text{P}_n(\text{x})=\frac{1}{\text{n}!\,2^{\text{n}}}\left\{\left(\mathbf{x}^2-1\right)^{\mathbf{n}}\right\}_{\mathbf{n}}=\frac{1}{\text{n}!\,2^{\text{n}}}\frac{\mathbf{d}^{\mathbf{n}}}{\mathbf{dx}^{\mathbf{n}}}\left(\mathbf{x}^2-1\right)^{\mathbf{n}}$$

REFERENCE:

- Higher Engineering Mathematics By B S Grewal
- Numerical Methods for Engineers and scientists by Joe D. Hoffman
- You tube channel **BECMAT16**
- Internet