Numerical Solution of Ordinary Differential Equations of First order and first degree

Introduction

Many ordinary differential equations can be solved by analytical methods discussed earlier giving closed form solutions i.e. expressing y in terms of a finite number of elementary functions of x. However, a majority of differential equations appearing in physical problems cannot be solved analytically. Thus it becomes imperative to discuss their solution by numerical methods.

Numerical methods for Initial value problem:

Consider the first order and first degree differential equations $\frac{dy}{dx} = f(x,y)$with the initial condition $y(x_0) = y_0$ that is $y = y_0$ and $x = x_0$ called initial value problem.

We discuss the following numerical methods for solving an initial value problem.

1. Taylor's series method
2. Modified Euler's method
3. Runge - Kutta method of order IV
4. Milne's Predictor - Corrector Method
5. Adams – Bashforth Predictor - Corrector Method

Type -1

Taylor's series method

Consider the first order and first degree differential equations $\frac{dy}{dx} = f(x,y)$ condition $y(x_0) = y_0$.

Taylor's series expansion of $y(x)$ in powers of $(x - x_0)$ is

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_1(x_0) + \frac{(x-x_0)^2}{2!}y_2(x_0) + \frac{(x-x_0)^3}{3!}y_3(x_0) + \frac{(x-x_0)^4}{4!}y_4(x_0) + \ldots\ldots$$

Where

$y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$, $y_3 = \frac{d^3y}{dx^3}$, $y_4 = \frac{d^4y}{dx^4}$, … …. at the point(x_0, y_0)

Worked Examples

1. Using Taylor's Series method, find the value of y at $x = 0.1$, and $x = 0.2$ for the initial value problem $\frac{dy}{dx} = 3x + y^2$, $y(0) = 1$.

Solution:

Taylor's Series expansion of $y(x)$ about a point x_0 is given by

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_1(x_0) + \frac{(x-x_0)^2}{2!}y_2(x_0) + \frac{(x-x_0)^3}{3!}y_3(x_0) + \frac{(x-x_0)^4}{4!}y_4(x_0) \ldots\ldots$$

Here, compare $y(x_0) = y_0 \Rightarrow y(0) = 1$, then $x_0 = 0$, $y_0 = 1$ and $y_1 = 3x + y^2$

$$y(x) = y_0 + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots \ldots\ldots\ldots\ldots(*)$$

$y_1 = 3x + y^2$; $y_1(0) = 3x_0 + {y_0}^2$; $y_1(0) = 3(0) + (1)^2$; [$y_1(0) = 1$]

Differentiate y_1 w.r.t x we get,

$y_2 = 3 + 2yy_1$; $y_2(0) = 3x_0 + 2y_0y_1(0)$; $y_2(0) = 3 + 2(1)(1)$; [$y_2(0) = 5$]

Differentiate y_2 w.r.t x we get,

$y_3 = 2(yy_{2+}y_1^2)$; $y_3(0) = 2(y_0y_2(0) + y_1^2(0)$; $y_3(0) = 2[(1)(5) + (1)^2]$; [$y_3(0) = 12$]

Differentiate y_3 w.r.t x we get,

$y_4 = 2(yy_3 + y_2y_1 + 2y_1y_2)$; $y_4(0) = 2(yy_3 + 3y_2y_1)$; $y_4(0) = 2[(1)(12) + 3(5)(1)]$; $y_4(0) = 2(12 + 15)$; $y_4(0) = 2(27)$; [$y_4(0) = 54$]

Substitute the values of $y_1(0)$, $y_2(0)$ $y_3(0)$ $y_4(0)$ in equation (*)

$$y(x) = 1 + \frac{x}{1!}1 + \frac{x^2}{2!}5 + \frac{x^3}{3!}12 + \frac{x^4}{4!}54$$

$$y(x) = 1 + x + \frac{5}{2}x^2 + 2x^3 + \frac{9}{4}x^4$$

This is called Taylors series expansion up to fourth degree term.

Put $x = 0.1$. $x = 0.2$

$$y(0.1) = 1 + 0.1 + \frac{5}{2}(0.1)^2 + 2(0.1)^3 + \frac{9}{4}(0.1)^4 = 1.12722$$

$$y(0.2) = 1 + 0.2 + \frac{5}{2}(0.2)^2 + 2(0.2)^3 + \frac{9}{4}(0.2)^4 = 1.3196$$

2. Using Taylor's Series method, find the value of y at $x = 0.1$, and $x = 0.2$ for the initial value problem $\frac{dy}{dx} = x^2y - 1,\ y(0) = 1$.

Solution:

Taylor's Series expansion of $y(x)$ about a point x_0 is given by

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_1(x_0) + \frac{(x-x_0)^2}{2!}y_2(x_0) + \frac{(x-x_0)^3}{3!}y_3(x_0) + \frac{(x-x_0)^4}{4!}y_4(x_0) \dots\dots$$

Here, compare $y(x_0) = y_0 \Rightarrow y(0) = 1$, then $x_0 = 0,\ y_0 = 1$ and $y_1 = x^2y - 1$

$$y(x) = y_0 + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots\dots\dots\dots\dots(*)$$

$y_1 = x^2y - 1;\quad y_1(0) = {x_0}^2y_0 - 1;\quad y_1(0) = (0^2)(1) - (1)^2;\qquad [\ y_1(0) = -1]$

Differentiate y_1 w.r.t x we get,

$y_2 = x^2y_1 + 2xy;\quad y_2(0) = {x_0}^2y_1(0) + 2x_0y_0;\quad y_2(0) = (0^2)(-1) + 2(0)(1);\ [\ y_2(0) = 0]$

Differentiate y_2 w.r.t x we get,

$$y_3 = x^2y_2 + 4xy_1 + 2y; y_3(0) = {x_0}^2y_2(0) + 4x_0y_1(0) + 2(1); y_3(0) = (0^2)(2) + 4(0)(-1) + 2(1);\ [\ y_3(0) = 2]$$

Differentiate y_3 w.r.t x we get,

$$y_4 = x^2y_3 + 6xy_2 + 6y_1; y_4(0) = {x_0}^2y_3(0) + 6x_0y_2(0) + 6y_1(0); y_4(0) = (0^2)(2) + 6(0)(2) + 6(-1);\ [\ y_4(0) = -6]$$

Substitute the values of $y_1(0)$, $y_2(0)$ $y_3(0)$ $y_4(0)$ in equation (*)

$$y(x) = 1 + \frac{x}{1!}(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6)$$

$$y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4}$$

This is called Taylors series expansion up to fourth degree term.

Put $x = 0.1$ and $x = 0.2$

$$y(0.1) = 1 - (0.1) + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} = 0.90031$$

$$y(0.2) = 1 - (0.2) + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} = 0.8023$$

3. Using Taylor's Series method, find the value of y at $x = 0.1$, and $x = 0.2$, for the initial value problem $\frac{dy}{dx} - 2y = 3e^x$, $y(0) = 0$.

Solution:

Taylor's Series expansion of $y(x)$ about a point x_0 is given by

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_1(x_0) + \frac{(x-x_0)^2}{2!}y_2(x_0) + \frac{(x-x_0)^3}{3!}y_3(x_0) + \frac{(x-x_0)^4}{4!}y_4(x_0) \ldots\ldots$$

Here, compare $y(x_0) = y_0 \Rightarrow y(0) = 0$, then $x_0 = 0$, $y_0 = 0$ and $y_1 = 2y + 3e^x$

$$y(x) = y_0 + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots \ldots\ldots\ldots\ldots(*)$$

$y_1 = 2y + 3e^x$; $y_1(0) = 2y_0 + 3e^x$; $y_1(0) = 2(0) + 3e^{(0)} = 0 + 3(1) = 3$; [$y_1(0) = 3$]

Differentiate y_2 w.r.t x we get,

$y_2 = 2y_1 + 3e^x$; $y_2(0) = 2y_1(0) + 3e^x$; $y_2(0) = 2(3) + 3e^{(0)} = 6 + 3(1 = 9$[$y_2(0) = 9$]

Differentiate y_3 w.r.t x we get,

$y_3 = 2y_2 + 3e^x$; $y_3(0) = 2y_2(0) + 3e^x$; $y_3(0) = 2(9) + 3e^{(0)} = 18 + 3(1)$
$= 21$; [$y_3(0) = 21$]

Differentiate y_3 w.r.t x we get,

$y_4 = 2y_3 + 3e^x$; $y_4(0) = 2y_3(0) + 3e^x$; $y_4(0) = 2(21) + 3e^{(0)} = 42 + 3(1)$
$= 45$; [$y_4(0) = 45$]

Substitute the values of $y_1(0)$, $y_2(0)$ $y_3(0)$ $y_4(0)$ in equation (*)

$$y(x) = 0 + \frac{x}{1!}(3) + \frac{x^2}{2!}(9) + \frac{x^3}{3!}(21) + \frac{x^4}{4!}(45)$$

$$y(x) = 3x + \frac{9x^2}{2} + \frac{7x^3}{2} + \frac{15x^4}{8}$$

This is called Taylors series expansion up to fourth degree term.

Put $x = 0.1$ and $x = 0.2$

$$y(0.1) = 3(0.1) + \frac{9(0.1)^2}{2} + \frac{7(0.1)^3}{2} + \frac{15(0.1)^4}{8} = 0.34869$$

$$y(0.2) = 3(0.2) + \frac{9(0.2)^2}{2} + \frac{7(0.2)^3}{2} + \frac{15(0.2)^4}{8} = 0.81100$$

4. Using Taylor's Series method, solve the initial value problem

$\frac{dy}{dx} = xy + 1$, $y(0) =$ 1. and hence find the value of y at $x = 0.1$

Solution:

Taylor's Series expansion of $y(x)$ about a point x_0 is given by

$$y(x) = y_0 + \frac{(x - x_0)}{1!}y_1(x_0) + \frac{(x - x_0)^2}{2!}y_2(x_0) + \frac{(x - x_0)^3}{3!}y_3(x_0) + \frac{(x - x_0)^4}{4!}y_4(x_0) \ldots\ldots$$

Here, compare $y(x_0) = y_0 \Rightarrow y(0) =$ 1, then $x_0 = 0$, $y_0 = 1$ and $y_1 = xy + 1$

$$y(x) = y_0 + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots \ldots\ldots\ldots\ldots(*)$$

$y_1 = xy + 1$; $y_1(0) = x_0y_0 + 1$; $y_1(0) = (0)(1) + 1$; [$y_1(0) = 1$]

Differentiate y_1 w.r.t x we get,

$y_2 = xy_1 + y$; $y_1(0) = x_0y_1(0) + 1$; $y_2(0) = (0)(1) + 1$; [$y_2(0) = 1$]

Differentiate y_2 w.r.t x we get,

$y_3 = xy_2 + 2y_1$; $y_3(0) = x_0y_2(0) + 2y_1(0)$; $y_2(0) = (0)(1) + 2(1)$; [$y_3(0) = 2$]

Differentiate y_3 w.r.t x we get,

$y_4 = xy_3 + 3y_2$; $y_1(0) = x_0 y_3(0) + 3y_2(0)$; $y_3(0) = (0)(2) + 3(1)$; $[\ y_4(0) = 3]$

Substitute the values of $y_1(0)$, $y_2(0)$ $y_3(0)$ $y_4(0)$ in equation (*)

$$y(x) = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(3)$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

This is called Taylors series expansion up to fourth degree term.

Put $x = 0.1$

$$y(0.1) = 1 + (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} = 1.105346$$

5. Using Taylor's Series method, solve the initial value problem

$\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$. and hence find the value of y at $x = 0.1$ and 0.2

Solution:

Taylor's Series expansion of $y(x)$ about a point x_0 is given by

$$y(x) = y_0 + \frac{(x - x_0)}{1!} y_1(x_0) + \frac{(x - x_0)^2}{2!} y_2(x_0) + \frac{(x - x_0)^3}{3!} y_3(x_0) + \frac{(x - x_0)^4}{4!} y_4(x_0) \dots\dots$$

Here, compare $y(x_0) = y_0 \Rightarrow y(0) = 1$, then $x_0 = 0$, $y_0 = 1$ and $y_1 = x^2 + y^2$

$$y(x) = y_0 + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \cdots \dots\dots\dots\dots (*)$$

$y_1 = x^2 + y^2$; $y_1(0) = {x^2}_0 + {y^2}_0$; $y_1(0) = 0^2 + 1^2 = 1$; $[\ y_1(0) = 1]$

Differentiate y_1 w.r.t x we get,

$y_2 = 2x + 2y\, y_1$; $y_2(0) = 2x_0 + 2\, y_0 y_1(0)$; $y_2(0) = 2(0) + 2(1)(1) = 2$; $[\ y_2(0) = 2]$

Differentiate y_2 w.r.t x we get,

$y_3 = 2 + 2y\, y_2 + 2{y_1}^2$; $y_3(0) = 2 + 2\, y_0 y_2(0)[\, y_1(0)]^2$; $y_3(0) = 2 + 2(1)(2) + 2(1)^2$
$= 8$; $[\ y_3(0) = 8]$

Differentiate y_3 w.r.t x we get,

$y_4 = 2y\,y_3 + 6\,y_1 y_2;\quad y_4(0) = 2\,y_0 y_3(0) + 6\,y_1(0)\,y_2(0);\quad y_4(0) = 2(1)(8) + 6(1)(2)$
$= 28\,;\,[\;y_4(0) = 28]$

Substitute the values of $y_1(0),\ y_2(0)\ y_3(0)\ y_4(0)$ in equation (1)

$$y(x) = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(8) + \frac{x^4}{4!}(28)$$

$$y(x) = 1 + x + x^2 + \frac{4x^3}{3} + \frac{7x^4}{6}$$

This is called Taylors series expansion up to fourth degree term.

Put $x = 0.1$ and $x = 0.2$

$$y(0.1) = 1 + (0.1) + (0.1)^2 + \frac{4(0.1)^3}{3} + \frac{7(0.1)^4}{6} = 1.1115$$

$$y(0.2) = 1 + (0.2) + (0.2)^2 + \frac{4(0.2)^3}{3} + \frac{7(0.2)^4}{6} = 1.2525$$

Type - 2

Modified Euler's method

Consider the initial value problem $\dfrac{dy}{dx} = f(x,y); y(x_0) = y_0$

Suppose we determine solution of this problem at a point $x_n = x_0 + nh$ (where h is step length) by using Euler's method

The solution is given by $y_n^p = y_{n-1} + hf(x_{n-1}, y_{n-1}), n = 1,2,3,\ldots$

Here, this will gives approximate solution by Euler's method. Since the accuracy is poor in this formula this value

Example. 1 Using modified Euler's method find y(0.2) by solving the equation

with h = 0.1 $\dfrac{dy}{dx} = x - y^2\,; y(0) = 1$

Solution:- By data

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$
y	$y_0 = 1$	$y_1 = ?$	$y_2 = ?$

$h = 0.1$ $\quad f(x,y) = x - y^2$

This problem has to be worked in two stages for finding y(0.2)

Stage 1:- First to calculate the y(0.1) y_1 From Euler's formula

$$y_1^p = y_0 + hf(x_0, y_0)$$
$$y_1^p = y_0 + h\left[x_0 - y_0^2\right]$$
$$y_1^p = 1 + 0.1\left[0 - (1)^2\right]$$
$$y_1^p = 0.9$$

By modified Euler's formula, we have

$$y_1^{c_1} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^p\right)\right]$$
$$y_1^{c_1} = y_0 + \frac{h}{2}\left[\left(x_0 - y_0^2\right) + \left(x_1 - \left(y_1^p\right)^2\right)\right]$$
$$y_1^{c_1} = 1 + \frac{0.1}{2}\left[-1 + \left(0.1 - (0.9)^2\right)\right]$$
$$y_1^{c_1} = 1 + 0.05\left[-0.9 - (0.9)^2\right] = 0.9145$$

The second Modified value of y_1

$$y_1^{c_2} = y_0 + \frac{h}{2}\left[\left(x_0 - y_0^2\right) + \left(x_1 - \left(y_1^{c_1}\right)^2\right)\right]$$
$$y_1^{c_2} = 1 + 0.05\left[-0.9 - (0.9145)^2\right] = 0.9132$$

The Third Modified value of y_1

$$y_1^{c_3} = y_0 + \frac{h}{2}\left[\left(x_0 - y_0^2\right) + \left(x_1 - \left(y_1^{c_2}\right)^2\right)\right]$$
$$y_1^{c_3} = 1 + 0.05\left[-0.9 - (0.9132)^2\right] = 0.9133$$
$$y_1 = y(0.1) = 0.9133$$

Type – 4

Predictor - Corrector Method

In the predictor – Corrector methods, Four prior values are required for finding the value of y at *x*. These Four values may be given or extract using the initial condition by Taylors series

A predictor formula is used to predict the value of y at x and then corrector formula is applied to improve this value.

We describe two such methods namely

1. Milne's Method

2. Adams Bashforth Method

Milne's Predictor –Corrector Method

Working rule:

Consider the initial value problem with a set of four points

$y(x_0) = y_0,\ y(x_1) = y_1, y(x_2) = y_2,\ y(x_3) = y_3,$ Here $x_0, x_1,\ x_2, x_3$ equally spaced. To find y4 at the point x4

Milne's Predictor formula

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

Milne's Corrector formula

$$y_4^c = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^p\right) \quad \text{where} \quad f_4^p = \frac{dy}{dx} = f(x_4, y_4^p)$$

To improve the accuracy again apply corrector formula by assuming

$$y_4^{c_1} = y_4^c$$

$$y_4^{c_1} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^c\right) \quad \text{where} \quad f_4^c = \frac{dy}{dx} = f(x_4, y_4^c)$$

Worked Examples

1. Using Milne's method, find y(0.8), given $y' = x - y^2$ given y(0) = 0, y(0.2) = 0.0200, y(0.4) = 0.0795, y(0.6) = 0.1762.

Solution:- Construct the table by using given values

x	y	$\frac{dy}{dx} = f(x,y) = x - y^2$
$x_0 = 0$	$y_0 = 0$	$f_0 = 0 - (0)^2 = 0$
$x_1 = 0.2$	$y_1 = 0.0200$	$f_1 = 0.2 - (0.0200)^2 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$f_2 = 0.4 - (0.0795)^2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$f_3 = 0.6 - (0.1799)^2 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	

By Milne's Predictor formula

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$y_4^p = 0 + \frac{4(0.2)}{3}\left[2(0.1996) - (0.3937) + 2(0.5689)\right]$$

$$y_4^p = 0.30488$$

By Milne's Corrector formula

$$y_4^c = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^p\right) \qquad f_4^p = x_4 - y_4^p = 0.7070$$

$$y_4^c = 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5689)f_3 + 0.7070\right]$$

$$y_4^c = 0.3045$$

To improve the accuracy of our results substitute the y_4^c in corrector formula Milne's Predictor formula

$$^{c_1}_{4} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^c\right) \quad f_4^c = x_4 - \left(y_4^c\right)^2 = 0.70723$$

$$y_4^{c_1} = 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5689) + 0.7072\right]$$

$$y_4^{c_1} = 0.3046$$

2. Compute y(0.4), by applying Milne's predictor corrector method. Use corrector formula twice for the differential equation. Given

$\frac{dy}{dx} = 2e^x - y$ and

x	0	0.1	0.2	0.3
y	2	2.010	2.04	2.09

Solution:- Construct the table by using given values

x	y	$\frac{dy}{dx} = f(x,y) = 2e^x - y$
$x_0 = 0$	$y_0 = 2$	$f_0 = 2e^0 - 2 = 0.0$
$x_1 = 0.1$	$y_1 = 2.010$	$f_1 = 2e^{0.1} - 2.010 = 0.20034$
$x_2 = 0.2$	$y_2 = 2.04$	$f_2 = 2e^{0.2} - 2.04 = 0.40281$
$x_3 = 0.3$	$y_3 = 2.09$	$f_3 = 2e^{0.3} - 2.09 = 0.60972$
$x_4 = 0.4$	$y_4 = ?$	

By Milne's Predictor formula

$$y_4^p = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$$

$$y_4^p = 2 + \frac{4(0.1)}{3}\left[2(0.20034) - (0.40281) + 2(0.60972)\right]$$

$$y_4^p = 2.16231$$

By Milne's Corrector formula

$$y_4^c = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^p\right) \qquad f_4^p = 2e^{x_4} - y_4^p = 0.82134$$

$$y_4^c = 2.04 + \frac{0.1}{3}\left[0.40281 + 4(0.60972) + 0.82134\right]$$

$$y_4^c = 2.1620$$

To improve the accuracy of our results substitute the y_4^c in corrector formula

$$y_4^{c_1} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^c\right)$$

$$f_4^c = 2e^{x_4} - y_4^c = 0.82155$$

$$y_4^{c_1} = 2.04 + \frac{0.1}{3}\left[0.40281 + 4(0.60972) + 0.82155\right]$$

$$y_4^{c_1} = 2.16211$$

It is the required value of *y* at *x* = 0.4

3. Find y at x = 0.3, using applying Milne's method. Given $\frac{dy}{dx} = \frac{x+y}{2}$ and

x	-0.1	0	0.1	0.2
y	0.90878	1	1.11145	1.25253

Solution:- Construct the table by using given values

x	y	$y' = \frac{dy}{dx} = f(x,y) = \frac{x+y}{2}$
$x_0 = -0.1$	$y_0 = 0.90878$	$f_0 = 0.40439$
$x_1 = 0$	$y_1 = 1.0000$	$f_1 = 0.5$
$x_2 = 0.1$	$y_2 = 1.11145$	$f_2 = 0.605725$
$x_3 = 0.2$	$y_3 = 1.25253$	$f_3 = 0.72626$
$x_4 = 0.3$	$y_4 = ?$	?

By Milne's Predictor formula

$$y_4^p = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$$

$$y_4^p = 0.90878 + \frac{4(0.1)}{3}\left[2(0.5) - (0.60572) + 2(0.72626)\right]$$

$$y_4^p = 1.15502$$

By Milne's Corrector formula

$$y_4^c = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^p\right)$$

$$f_4^p = \frac{x_4 + y_4^p}{2} = 0.72751$$

$$y_4^c = 1.11145 + \frac{0.1}{3}\left[0.60572 + 4(0.72626) + 0.72751\right]$$

$$y_4^c = 1.25272$$

To improve the accuracy of our results substitute the y_4^c in corrector formula

$$y_4^{c_1} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^c\right)$$

$$f_4^c = \frac{x_4 + y_4^c}{2} = 0.77636$$

$$y_4^{c_1} = 1.11145 + \frac{0.1}{3}\left[0.60572 + 4(0.72626) + 0.77636\right]$$

$$y_4^{c_1} = 1.25435$$

It is the required value of *y* at *x* = 0.3

4. Find y at x = 4.4 using Milne's method. Given

x	4.0	4.1	4.2	4.3
y	1	1.0049	1.0097	1.0143

$5xy' - 2 + y^2 = 0$ and

Solution:- Construct the table by using given values

x	y	$y' = \frac{dy}{dx} = f(x,y) = \frac{2-y^2}{5x}$
$x_0 = 4$	$y_0 = 1$	$f_0 = \frac{2-{y_0}^2}{5x_0} = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$f_1 = \frac{2-{y_1}^2}{5x_1} = 0.0485$
$x_2 = 4.2$	$y_2 = 1.0097$	$f_2 = \frac{2-{y_2}^2}{5x_2} = 0.0467$
$x_3 = 4.3$	$y_3 = 1.0143$	$f_3 = \frac{2-{y_3}^2}{5x_3} = 0.0452$
$x_4 = 4.4$	$y_4 = ?$	?

By Milne's Predictor formula

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$y_4^p = 1 + \frac{4(0.1)}{3}[2(0.0485) - (0.0467) + 2(0.0452)]$$

$$y_4^p = 1.01876$$

By Milne's Corrector formula

$$y_4^c = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^p\right)$$

$$f_4^p = \frac{2-\left(y_4^p\right)^2}{5x_4} = 0.04373$$

$$y_4^c = 1.0097 + \frac{0.1}{3}[0.0467 + 4(0.0456) + 0.04373]$$

$$y_4^c = 1.00909$$

To improve the accuracy of our results substitute the y_4^c in corrector formula

$$y_4^{c_1} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^c\right)$$

$$f_4^p = \frac{2-\left(y_4^c\right)^2}{5x_4} = 0.04462$$

$$y_4^{c_1} = 1.0097 + \frac{0.1}{3}\left[0.0467 + 4(0.0452) + 0.04462\right]$$

$$y_4^{c_1} = 1.01877$$

It is the required value of *y* at *x* = 4.4

5. Find y at x = 0.4 using Milne's method. Given

$$y' = xy + y^2,\ \ y(0)=1, y(0.1)=1.1169,$$
$$y(0.2)=1.2773, y(0.3)=1.5049$$

Solution: - Construct the table by using given values

x	y	$y' = f(x,y) = xy + y^2$
$x_0 = 0$	$y_0 = 1$	$f_0 = x_0y_0 + y_0^2 = 1$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = x_1y_1 + y_1^2 = 1.3592$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = x_2y_2 + y_2^2 = 1.887$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = x_3y_3 + y_3^2 = 2.7162$
$x_4 = 0.4$	$y_4 = ?$	?

By Milne's Predictor formula

$$y_4^p = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$$

$$y_4^p = 1 + \frac{4(0.1)}{3}\left[2(1.3592) - (1.887) + 2(2.7162)\right]$$

$$y_4^p = 1.8352$$

By Milne's Corrector formula

$$y_4^c = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^p\right)$$

$$f_4^p = x_4 y_4 + \left(y_4^p\right)^2 = 4.1020$$

$$y_4^c = 1.2773 + \frac{0.1}{3}\left[1.887 + 4(2.7162) + 4.102\right] = 1.8391$$

To improve the accuracy of our results substitute the y_4^c in corrector formula $y_4^{c_1} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^c\right)$ $\quad f_4^c = x_4 y_4 + \left(y_4^c\right)^2 = 4.1179$

$$y_4^{c_1} = 1.2773 + \frac{0.1}{3}\left[1.887 + 4(2.7162) + 4.1179\right]$$

$$y_4^{c_1} = 1.8396$$

It is the required value of y at x = 0.4

II . Adams Bashforth Predictor Corrector Method

Working rule:

Consider the initial value problem with a set of four points

$y(x_0) = y_0,\ y(x_1) = y_1, y(x_2) = y_2,\ y(x_3) = y_3,$ Here x_0, x_1, x_2, x_3 equally spaced. To find y4 at the point x4

Adams-Bashforth Predictor formula

$$y_4^p = y_3 + \frac{h}{24}\left(55f_3 - 59f_2 + 37f_1 - 9f_0\right)$$

Adams-Bashforth Corrector formula

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^p\right)$$

Where $f_4^p = \frac{dy}{dx} = f(x_4, y_4^p)$

To improve the accuracy again apply corrector formula by assuming

$$y_4^{c_1} = y_4^c$$

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^c\right)$$

Where $f_4^c = \frac{dy}{dx} = f(x_4, y_4^c)$

Example: 1

Find y(0.4), by applying Adams-Bashforth method given that $y' = \frac{xy}{2}$ and

x	0	0.1	0.2	0.3
y	1	1.0025	1.0101	1.0228

Solution:- Construct the table by using given values

x	y	$\frac{dy}{dx} = f(x,y) = \frac{xy}{2}$
$x_0 = 0$	$y_0 = 1$	$f_0 = 0$
$x_1 = 0.1$	$y_1 = 1.0025$	$f_1 = 0.0501$
$x_2 = 0.2$	$y_2 = 1.0101$	$f_2 = 0.1010$
$x_3 = 0.3$	$y_3 = 0.1762$	$f_3 = 0.1534$
$x_4 = 0.4$	$y_4 = ?$	

Adams-Bashforth Predictor formula

$$y_4^p = y_3 + \frac{h}{24}\left(55f_3 - 59f_2 + 37f_1 - 9f_0\right)$$

$$y_4^p = 1.0228 + \frac{0.1}{24}\left[55(0.1534) - 59(0.1010) + 37(0.05012)_1 - 9(0)\right]$$

$$y_4^p = 1.0408$$

Adams-Bashforth Corrector formula

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^p\right)$$

$$f_4^p = \frac{x_4 y_4^p}{2} = \frac{(0.4)(1.0408)}{2} = 0.2081$$

$$y_4^c = 1.0228 + \frac{0.1}{24}\left[0.0501 - 5(0.1010) + 19(0.1534) + 9(0.2081)\right]$$

$$y_4^c = 1.0408$$

Example. 2 Given $y' = x^2(1+y)$, $y(1)=1, y(1.1)=1.233, y(1.2)=1.548, y(1.3)=1.979$ determine $y(1.4)$ by Adams- Bashforth method

Solution:- Construct the table by using given values

x	y	$y' = f(x,y) = x^2(1+y)$
$x_0 = 1$	$y_0 = 1$	$f_0 = x_0^2(1+y_0) = 2$
$x_1 = 1.1$	$y_1 = 1.233$	$f_1 = x_1^2(1+y_1) = 2.702$
$x_2 = 1.2$	$y_2 = 1.548$	$f_2 = x_2^2(1+y_2) = 3.669$
$x_3 = 1.3$	$y_3 = 1.979$	$f_3 = x_3^2(1+y_3) = 5.035$
$x_4 = 1.4$	$y_4 = ?$	?

Adams-Bashforth Predictor formula

$$y_4^p = y_3 + \frac{h}{24}\left(55f_3 - 59f_2 + 37f_1 - 9f_0\right)$$

$$y_4^p = 1.979 + \frac{0.1}{24}\left[55(5.035) - 59(3.669) + 37(2.702) - 9(2)\right]$$

$$y_4^p = 2.572$$

Adams-Bashforth Corrector formula

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^p\right)$$

$$f_4^p = x_4^2(1+y_4^p) = (1.4)^2(1+2.572) = 7.001$$

$$y_4^c = 1.979 + \frac{0.1}{24}\left[2.702 - 5(3.669) + 19(5.035) + 9(7.001)\right]$$

$$y_4^c = 2.575$$

To correct this solution again apply Adams-Bashforth Corrector formula,

Substitute y_4^c in $y_4^{c_1}$

$$y_4^{c_1} = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^c\right)$$

$$f_4^c = x_4^2(1+y_4^c) = (1.4)^2(1+2.575) = 7.007$$

$$y_4^{c_1} = 1.979 + \frac{0.1}{24}\left[2.702 - 5(3.669) + 19(5.035) + 9(7.007)\right]$$

$$y_4^{c_1} = 2.575$$

Example. 3 Given $\frac{dy}{dx} = 2e^x y$, $y(0)=2, y(0.1)=2.4725$, $y(0.2)=3.1261$, $y(0.3)=4.0524$ determine y(0.4) by Adams- Bashforth method.

Solution:- Construct the table by using given values

x	y	$y' = f(x,y) = 2e^x y)$
$x_0 = 0$	$y_0 = 1$	$f_0 = 4$
$x_1 = 0.1$	$y_1 = 2.4725$	$f_1 = 5.4652$
$x_2 = 0.2$	$y_2 = 3.1261$	$f_2 = 7.6364$
$x_3 = 0.3$	$y_3 = 4.0524$	$f_3 = 10.9406$
$x_4 = 0.4$	$y_4 = ?$	?

Adams-Bashforth Predictor formula

$$y_4^p = y_3 + \frac{h}{24}\left(55f_3 - 59f_2 + 37f_1 - 9f_0\right)$$

$$y_4^p = 4.0524 + \frac{0.1}{24}\left[55(10.9406) - 59(7.6364) + 37(5.4652) - 9(4)\right]$$

$$y_4^p = 5.3749$$

Adams-Bashforth Corrector formula

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^p\right)$$

$$f_4^p = 2y_4^p e^{x_4} = 2(5.3749)e^{0.4} = 16.0366$$

$$y_4^c = 4.0524 + \frac{0.1}{24}\left[5.4652 - 5(7.6364) + 19(10.9406) + 9(16.0366)\right]$$

$$y_4^c = 5.3835$$

To correct this solution again apply Adams-Bashforth Corrector formula,

Substitute y_4^c in $y_4^{c_1}$

$$y_4^{c_1} = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^c\right)$$

$$f_4^c = 2y_4^c e^{x_4} = 2(5.33835)e^{0.4} = 16.06248$$

$$y_4^{c_1} = 4.0524 + \frac{0.1}{24}\left[5.4652 - 5(7.6364) + 19(10.9406) + 9(16.0624)\right]$$

$$y_4^{c_1} = 5.3845$$

Example. 4

Solve the differential equation

$\frac{dy}{dx} = x - y^2$, at $x = 0.8$ given $y(0)=0, y(0.2)=0.02, y(0.4)=0.0795, y(0.6)=0.1762$

Using Adams- Bashforth method

Solution:- Construct the table by using given values

x	y	$\frac{dy}{dx} = f(x,y) = x - y^2$
$x_0 = 0$	$y_0 = 0$	$f_0 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$f_1 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$f_2 = 0.3936$
$x_3 = 0.6$	$y_3 = 0.1762$	$f_3 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	?

Adams-Bashforth Predictor formula

$$y_4^p = y_3 + \frac{h}{24}(55 f_3 - 59 f_2 + 37 f_1 - 9 f_0)$$

$$y_4^p = 0.1762 + \frac{0.2}{24}[55(0.5689) - 59(0.3936) + 37(0.1996) - 9(0)]$$

$$y_4^p = 0.30495$$

Adams-Bashforth Corrector formula

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5 f_2 + 19 f_3 + 9 f_4^p\right)$$

$$f_4^p = x_4 - \left(y_4^p\right)^2 = 0.8 - (0.3049)^2 = 0.70701$$

$$y_4^c = 0.1762 + \frac{0.2}{24}[0.1996 - 5(0.3936) + 19(0.56895) + 9(0.70701)]$$

$$y_4^c = 0.30457$$

To correct this solution again apply Adams-Bashforth Corrector formula,

Substitute y_4^c in $y_4^{c_1}$

$$y_4^{c_1} = y_3 + \frac{h}{24}\left(f_1 - 5 f_2 + 19 f_3 + 9 f_4^c\right)$$

$$f_4^c = x_4 - \left(y_4^p\right)^2 = 0.8 - (0.30457)^2 = 0.70724$$

$$y_4^c = 0.1762 + \frac{0.2}{24}[0.1996 - 5(0.3936) + 19(0.56895) + 9(0.70724)]$$

$$y_4^c = 0.30459$$

Example. 5

Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2}{1+y^2}, \quad \text{at } x = 1.0 \text{ given } y(0)=1, y(0.25)=1.0026,$$

$y(0.5)=1.0206, \; y(0.75)=1.0679$

Solution:- Construct the table by using given values

x	y	$\frac{dy}{dx} = f(x,y) = \frac{x^2}{1+y^2}$
$x_0 = 0$	$y_0 = 1$	$f_0 = 1$
$x_1 = 0.25$	$y_1 = 1.0026$	$f_1 = 0.0312$
$x_2 = 0.5$	$y_2 = 1.0206$	$f_2 = 0.1225$
$x_3 = 0.75$	$y_3 = 1.0679$	$f_3 = 0.2628$
$x_4 = 1.0$	$y_4 = ?$	?

Adams-Bashforth Predictor formula

$$y_4^p = y_3 + \frac{h}{24}\left(55 f_3 - 59 f_2 + 37 f_1 - 9 f_0\right)$$

$$y_4^p = 1.0679 + \frac{0.25}{24}\left[55(0.2628) - 59(0.1225) + 37(0.0312) - 9(0)\right]$$

$$y_4^p = 1.1552$$

Adams-Bashforth Corrector formula

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5 f_2 + 19 f_3 + 9 f_4^p\right)$$

$$f_4^p = \frac{x_4^2}{1+\left(y_4^p\right)^2} = \frac{1^2}{1+(1.1552)^2} = 0.4284$$

$$y_4^c = 1.0679 + \frac{0.25}{24}\left[0.0312 - 5(0.1224) + 19(0.2628) + 9(0.4284)\right]$$

$$y_4^c = 1.154$$

To correct this solution again apply Adams-Bashforth Corrector formula,

Substitute y_4^c in $y_4^{c_1}$

$$y_4^{c_1} = y_3 + \frac{h}{24}\left(f_1 - 5 f_2 + 19 f_3 + 9 f_4^c\right)$$

$$f_4^c = \frac{x_4^2}{1+\left(y_4^c\right)^2} = \frac{1^2}{1+(1.154)^2} = 0.4289$$

$$y_4^c = 1.0679 + \frac{0.25}{24}\left[0.0312 - 5(0.1224) + 19(0.2628) + 9(0.4289)\right]$$

$$y_4^c = 1.1541$$

Numerical Solution of Ordinary Differential Equations of First order and first degree

Introduction

Many ordinary differential equations can be solved by analytical methods discussed earlier giving closed form solutions i.e. expressing y in terms of a finite number of elementary functions of x. However, a majority of differential equations appearing in physical problems cannot be solved analytically. Thus it becomes imperative to discuss their solution by numerical methods.

Numerical methods for Initial value problem:

Consider the first order and first degree differential equations $\frac{dy}{dx} = f(x,y)$with the initial condition $y(x_0) = y_0$ that is $y = y_0$ and $x = x_0$ called initial value problem.

We discuss the following numerical methods for solving an initial value problem.

1. Taylor's series method
2. Modified Euler's method
3. Runge - Kutta method of order IV
4. Milne's Predictor - Corrector Method
5. Adams – Bashforth Predictor - Corrector Method

Type -1

Taylor's series method

Consider the first order and first degree differential equations $\frac{dy}{dx} = f(x,y)$ condition $y(x_0) = y_0$.

Taylor's series expansion of $y(x)$ in powers of $(x - x_0)$ is

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_1(x_0) + \frac{(x-x_0)^2}{2!}y_2(x_0) + \frac{(x-x_0)^3}{3!}y_3(x_0) + \frac{(x-x_0)^4}{4!}y_4(x_0) + \ldots\ldots$$

Where

$y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$, $y_3 = \frac{d^3y}{dx^3}$, $y_4 = \frac{d^4y}{dx^4}$, … …. at the point(x_0, y_0)

Worked Examples

1. Using Taylor's Series method, find the value of y at $x = 0.1$, and $x = 0.2$ for the initial value problem $\frac{dy}{dx} = 3x + y^2$, $y(0) = 1$.

Solution:

Taylor's Series expansion of $y(x)$ about a point x_0 is given by

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_1(x_0) + \frac{(x-x_0)^2}{2!}y_2(x_0) + \frac{(x-x_0)^3}{3!}y_3(x_0) + \frac{(x-x_0)^4}{4!}y_4(x_0) \ldots\ldots$$

Here, compare $y(x_0) = y_0 \Rightarrow y(0) = 1$, then $x_0 = 0$, $y_0 = 1$ and $y_1 = 3x + y^2$

$$y(x) = y_0 + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots \ldots\ldots\ldots\ldots (*)$$

$y_1 = 3x + y^2$; $y_1(0) = 3x_0 + {y_0}^2$; $y_1(0) = 3(0) + (1)^2$; [$y_1(0) = 1$]

Differentiate y_1 w.r.t x we get,

$y_2 = 3 + 2yy_1$; $y_2(0) = 3x_0 + 2y_0y_1(0)$; $y_2(0) = 3 + 2(1)(1)$; [$y_2(0) = 5$]

Differentiate y_2 w.r.t x we get,

$y_3 = 2(yy_{2+}y_1^2)$; $y_3(0) = 2(y_0y_2(0) + y_1^2(0)$; $y_3(0) = 2[(1)(5) + (1)^2]$; [$y_3(0) = 12$]

Differentiate y_3 w.r.t x we get,

$y_4 = 2(yy_3 + y_2y_1 + 2y_1y_2)$; $y_4(0) = 2(yy_3 + 3y_2y_1)$; $y_4(0) = 2[(1)(12) + 3(5)(1)]$; $y_4(0) = 2(12 + 15)$; $y_4(0) = 2(27)$; [$y_4(0) = 54$]

Substitute the values of $y_1(0)$, $y_2(0)$ $y_3(0)$ $y_4(0)$ in equation (*)

$$y(x) = 1 + \frac{x}{1!}1 + \frac{x^2}{2!}5 + \frac{x^3}{3!}12 + \frac{x^4}{4!}54$$

$$y(x) = 1 + x + \frac{5}{2}x^2 + 2x^3 + \frac{9}{4}x^4$$

This is called Taylors series expansion up to fourth degree term.

Put $x = 0.1$. $x = 0.2$

$$y(0.1) = 1 + 0.1 + \frac{5}{2}(0.1)^2 + 2(0.1)^3 + \frac{9}{4}(0.1)^4 = 1.12722$$

$$y(0.2) = 1 + 0.2 + \frac{5}{2}(0.2)^2 + 2(0.2)^3 + \frac{9}{4}(0.2)^4 = 1.3196$$

2. Using Taylor's Series method, find the value of y at $x = 0.1$, and $x = 0.2$ for the initial value problem $\frac{dy}{dx} = x^2y - 1,\ y(0) = 1.$

Solution:

Taylor's Series expansion of $y(x)$ about a point x_0 is given by

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_1(x_0) + \frac{(x-x_0)^2}{2!}y_2(x_0) + \frac{(x-x_0)^3}{3!}y_3(x_0) + \frac{(x-x_0)^4}{4!}y_4(x_0) \ldots\ldots$$

Here, compare $y(x_0) = y_0 \Rightarrow y(0) = 1$, then $x_0 = 0$, $y_0 = 1$ and $y_1 = x^2y - 1$

$$y(x) = y_0 + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots \ldots\ldots\ldots\ldots(*)$$

$y_1 = x^2y - 1;\quad y_1(0) = {x_0}^2y_0 - 1;\quad y_1(0) = (0^2)(1) - (1)^2;\qquad [\ y_1(0) = -1]$

Differentiate y_1 w.r.t x we get,

$y_2 = x^2y_1 + 2xy;\quad y_2(0) = {x_0}^2y_1(0) + 2x_0y_0;\quad y_2(0) = (0^2)(-1) + 2(0)(1);\ [\ y_2(0) = 0]$

Differentiate y_2 w.r.t x we get,

$y_3 = x^2y_2 + 4xy_1 + 2y; y_3(0) = {x_0}^2y_2(0) + 4x_0y_1(0) + 2(1); y_3(0)$
$= (0^2)(2) + 4(0)(-1) + 2(1);\ [\ y_3(0) = 2]$

Differentiate y_3 w.r.t x we get,

$y_4 = x^2y_3 + 6xy_2 + 6y_1; y_4(0) = {x_0}^2y_3(0) + 6x_0y_2(0) + 6y_1(0); y_4(0)$
$= (0^2)(2) + 6(0)(2) + 6(-1);\ [\ y_4(0) = -6]$

Substitute the values of $y_1(0),\ y_2(0)\ y_3(0)\ y_4(0)$ in equation (*)

$$y(x) = 1 + \frac{x}{1!}(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6)$$

$$y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4}$$

This is called Taylors series expansion up to fourth degree term.

Put $x = 0.1$ and $x = 0.2$

$$y(0.1) = 1 - (0.1) + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} = 0.90031$$

$$y(0.2) = 1 - (0.2) + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} = 0.8023$$

3. Using Taylor's Series method, find the value of y at $x = 0.1$, and $x = 0.2$, for the initial value problem $\frac{dy}{dx} - 2y = 3e^x,\ y(0) = 0.$

Solution:

Taylor's Series expansion of $y(x)$ about a point x_0 is given by

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_1(x_0) + \frac{(x-x_0)^2}{2!}y_2(x_0) + \frac{(x-x_0)^3}{3!}y_3(x_0) + \frac{(x-x_0)^4}{4!}y_4(x_0) \ldots\ldots$$

Here, compare $y(x_0) = y_0 \Rightarrow y(0) = 0$, then $x_0 = 0,\ y_0 = 0$ and $y_1 = 2y + 3e^x$

$$y(x) = y_0 + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots \ldots\ldots\ldots\ldots (*)$$

$y_1 = 2y + 3e^x;\ \ y_1(0) = 2y_0 + 3e^x;\ \ y_1(0) = 2(0) + 3e^{(0)} = 0 + 3(1) = 3\ ;\ [\ y_1(0) = 3]$

Differentiate y_2 w.r.t x we get,

$y_2 = 2y_1 + 3e^x;\ \ y_2(0) = 2y_1(0) + 3e^x;\ y_2(0) = 2(3) + 3e^{(0)} = 6 + 3(1 = 9[\ y_2(0) = 9]$

Differentiate y_3 w.r.t x we get,

$y_3 = 2y_2 + 3e^x;\ \ y_3(0) = 2y_2(0) + 3e^x;\ y_3(0) = 2(9) + 3e^{(0)} = 18 + 3(1)$
$= 21;[\ y_3(0) = 21]$

Differentiate y_3 w.r.t x we get,

$y_4 = 2y_3 + 3e^x$; $y_4(0) = 2y_3(0) + 3e^x$; $y_4(0) = 2(21) + 3e^{(0)} = 42 + 3(1)$
$= 45$; [$y_4(0) = 45$]

Substitute the values of $y_1(0)$, $y_2(0)$ $y_3(0)$ $y_4(0)$ in equation (*)

$$y(x) = 0 + \frac{x}{1!}(3) + \frac{x^2}{2!}(9) + \frac{x^3}{3!}(21) + \frac{x^4}{4!}(45)$$

$$y(x) = 3x + \frac{9x^2}{2} + \frac{7x^3}{2} + \frac{15x^4}{8}$$

This is called Taylors series expansion up to fourth degree term.

Put $x = 0.1$ and $x = 0.2$

$$y(0.1) = 3(0.1) + \frac{9(0.1)^2}{2} + \frac{7(0.1)^3}{2} + \frac{15(0.1)^4}{8} = 0.34869$$

$$y(0.2) = 3(0.2) + \frac{9(0.2)^2}{2} + \frac{7(0.2)^3}{2} + \frac{15(0.2)^4}{8} = 0.81100$$

4. Using Taylor's Series method, solve the initial value problem

$\frac{dy}{dx} = xy + 1$, $y(0) = 1$. and hence find the value of y at $x = 0.1$

Solution:

Taylor's Series expansion of $y(x)$ about a point x_0 is given by

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_1(x_0) + \frac{(x-x_0)^2}{2!}y_2(x_0) + \frac{(x-x_0)^3}{3!}y_3(x_0) + \frac{(x-x_0)^4}{4!}y_4(x_0) \ldots\ldots$$

Here, compare $y(x_0) = y_0 \Rightarrow y(0) = 1$, then $x_0 = 0$, $y_0 = 1$ and $y_1 = xy + 1$

$$y(x) = y_0 + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots \ldots\ldots\ldots\ldots(*)$$

$y_1 = xy + 1$; $y_1(0) = x_0y_0 + 1$; $y_1(0) = (0)(1) + 1$; [$y_1(0) = 1$]

Differentiate y_1 w.r.t x we get,

$y_2 = xy_1 + y$; $y_1(0) = x_0y_1(0) + 1$; $y_2(0) = (0)(1) + 1$; [$y_2(0) = 1$]

Differentiate y_2 w.r.t x we get,

$y_3 = xy_2 + 2y_1$; $y_3(0) = x_0y_2(0) + 2y_1(0)$; $y_2(0) = (0)(1) + 2(1)$; [$y_3(0) = 2$]

Differentiate y_3 w.r.t x we get,

$y_4 = xy_3 + 3y_2$; $y_1(0) = x_0y_3(0) + 3y_2(0)$; $y_3(0) =\ (0)(2) + 3(1)$;   [$y_4(0) = 3$]

Substitute the values of $y_1(0)$, $y_2(0)$ $y_3(0)$ $y_4(0)$ in equation (*)

$$y(x) = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(3)$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

This is called Taylors series expansion up to fourth degree term.

Put $x = 0.1$

$$y(0.1) = 1 + (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} = 1.105346$$

5. Using Taylor's Series method, solve the initial value problem

$\frac{dy}{dx} = x^2 + y^2$, $y(0) =\ 1$. and hence find the value of y at $x = 0.1$ and 0.2

Solution:

Taylor's Series expansion of $y(x)$ about a point x_0 is given by

$$y(x) = y_0 + \frac{(x - x_0)}{1!}y_1(x_0) + \frac{(x - x_0)^2}{2!}y_2(x_0) + \frac{(x - x_0)^3}{3!}y_3(x_0) + \frac{(x - x_0)^4}{4!}y_4(x_0) \dots\dots$$

Here, compare $y(x_0) = y_0 \Rightarrow y(0) =\ 1$, then $x_0 = 0$, $y_0 = 1$ and $y_1 = x^2 + y^2$

$$y(x) = y_0 + \frac{x}{1!}y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots \dots\dots\dots\dots (*)$$

$y_1 = x^2 + y^2$; $y_1(0) = {x^2}_0 + {y^2}_0$; $y_1(0) = 0^2 + 1^2 = 1$;   [$y_1(0) = 1$]

Differentiate y_1 w.r.t x we get,

$y_2 = 2x + 2y\, y_1$; $y_2(0) = 2x_0 + 2\, y_0y_1(0)$; $y_2(0) = 2(0) + 2(1)(1) = 2$;   [$y_2(0) = 2$]

Differentiate y_2 w.r.t x we get,

$y_3 = 2 + 2y\, y_2 + 2{y_1}^2$; $y_3(0) = 2 + 2\, y_0y_2(0)[\, y_1(0)]^2$; $y_3(0) = 2 + 2(1)(2) + 2(1)^2$
$= 8$; [$y_3(0) = 8$]

Differentiate y_3 w.r.t x we get,

$y_4 = 2y\, y_3 + 6\, y_1 y_2$; $y_4(0) = 2\, y_0 y_3(0) + 6\, y_1(0)\, y_2(0)$; $y_4(0) = 2(1)(8) + 6(1)(2)$
$= 28$; [$y_4(0) = 28$]

Substitute the values of $y_1(0)$, $y_2(0)$ $y_3(0)$ $y_4(0)$ in equation (1)

$$y(x) = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(8) + \frac{x^4}{4!}(28)$$

$$y(x) = 1 + x + x^2 + \frac{4x^3}{3} + \frac{7x^4}{6}$$

This is called Taylors series expansion up to fourth degree term.

Put $x = 0.1$ and $x = 0.2$

$$y(0.1) = 1 + (0.1) + (0.1)^2 + \frac{4(0.1)^3}{3} + \frac{7(0.1)^4}{6} = 1.1115$$

$$y(0.2) = 1 + (0.2) + (0.2)^2 + \frac{4(0.2)^3}{3} + \frac{7(0.2)^4}{6} = 1.2525$$

Type - 2

Modified Euler's method

Consider the initial value problem $\frac{dy}{dx} = f(x, y); y(x_0) = y_0$

Suppose we determine solution of this problem at a point $x_n = x_0 + nh$ (where h is step length) by using Euler's method

The solution is given by $y_n^p = y_{n-1} + hf(x_{n-1}, y_{n-1}), n = 1,2,3,...$

Here, this will gives approximate solution by Euler's method. Since the accuracy is poor in this formula this value

Example. 1 Using modified Euler's method find y(0.2) by solving the equation

with h = 0.1 $\frac{dy}{dx} = x - y^2 ; y(0) = 1$

Solution:- By data

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$
y	$y_0 = 1$	$y_1 = ?$	$y_2 = ?$

$h = 0.1$ $f(x, y) = x - y^2$

This problem has to be worked in two stages for finding y(0.2)

Stage 1:- First to calculate the y(0.1) y_1 From Euler's formula

$$y_1^p = y_0 + hf(x_0, y_0)$$
$$y_1^p = y_0 + h\left[x_0 - y_0^2\right]$$
$$y_1^p = 1 + 0.1\left[0 - (1)^2\right]$$
$$y_1^p = 0.9$$

By modified Euler's formula, we have

$$y_1^{c_1} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^p\right)\right]$$
$$y_1^{c_1} = y_0 + \frac{h}{2}\left[\left(x_0 - y_0^2\right) + \left(x_1 - \left(y_1^p\right)^2\right)\right]$$
$$y_1^{c_1} = 1 + \frac{0.1}{2}\left[-1 + \left(0.1 - (0.9)^2\right)\right]$$
$$y_1^{c_1} = 1 + 0.05\left[-0.9 - (0.9)^2\right] = 0.9145$$

The second Modified value of y_1

$$y_1^{c_2} = y_0 + \frac{h}{2}\left[\left(x_0 - y_0^2\right) + \left(x_1 - \left(y_1^{c_1}\right)^2\right)\right]$$
$$y_1^{c_2} = 1 + 0.05\left[-0.9 - (0.9145)^2\right] = 0.9132$$

The Third Modified value of y_1

$$y_1^{c_3} = y_0 + \frac{h}{2}\left[\left(x_0 - y_0^2\right) + \left(x_1 - \left(y_1^{c_2}\right)^2\right)\right]$$
$$y_1^{c_3} = 1 + 0.05\left[-0.9 - (0.9132)^2\right] = 0.9133$$
$$y_1 = y(0.1) = 0.9133$$

Type – 4

Predictor - Corrector Method

In the predictor – Corrector methods, Four prior values are required for finding the value of y at *x*. These Four values may be given or extract using the initial condition by Taylors series

A predictor formula is used to predict the value of y at x and then corrector formula is applied to improve this value.

We describe two such methods namely

1. Milne's Method

2. Adams Bashforth Method

Milne's Predictor –Corrector Method

Working rule:

Consider the initial value problem with a set of four points

$y(x_0)=y_0,\ y(x_1)=y_1, y(x_2)=y_2,\ y(x_3)=y_3,$ Here $x_0, x_1,\ x_2, x_3$ equally spaced. To find y4 at the point x4

Milne's Predictor formula

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

Milne's Corrector formula

$$y_4^c = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^p\right) \quad \text{where} \quad f_4^p = \frac{dy}{dx} = f(x_4, y_4^p)$$

To improve the accuracy again apply corrector formula by assuming

$$y_4^{c_1} = y_4^c$$

$$y_4^{c_1} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^c\right) \quad \text{where} \quad f_4^c = \frac{dy}{dx} = f(x_4, y_4^c)$$

Worked Examples

1. Using Milne's method, find y(0.8), given $y' = x - y^2$ given y(0) = 0, y(0.2) = 0.0200, y(0.4) = 0.0795, y(0.6) = 0.1762.

Solution:- Construct the table by using given values

x	y	$\frac{dy}{dx} = f(x,y) = x - y^2$
$x_0 = 0$	$y_0 = 0$	$f_0 = 0 - (0)^2 = 0$
$x_1 = 0.2$	$y_1 = 0.0200$	$f_1 = 0.2 - (0.0200)^2 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$f_2 = 0.4 - (0.0795)^2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$f_3 = 0.6 - (0.1799)^2 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	

By Milne's Predictor formula

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$y_4^p = 0 + \frac{4(0.2)}{3}[2(0.1996) - (0.3937) + 2(0.5689)]$$

$$y_4^p = 0.30488$$

By Milne's Corrector formula

$$y_4^c = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^p\right) \qquad f_4^p = x_4 - y_4^p = 0.7070$$

$$y_4^c = 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5689)f_3 + 0.7070\right]$$

$$y_4^c = 0.3045$$

To improve the accuracy of our results substitute the y_4^c in corrector formula Milne's Predictor formula

$$_4^{c_1} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^c\right) \quad f_4^c = x_4 - \left(y_4^c\right)^2 = 0.70723$$

$$y_4^{c_1} = 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5689) + 0.7072\right]$$

$$y_4^{c_1} = 0.3046$$

2. Compute y(0.4), by applying Milne's predictor corrector method. Use corrector formula twice for the differential equation. Given

$\frac{dy}{dx} = 2e^x - y$ and

x	0	0.1	0.2	0.3
y	2	2.010	2.04	2.09

Solution:- Construct the table by using given values

x	y	$\frac{dy}{dx} = f(x,y) = 2e^x - y$
$x_0 = 0$	$y_0 = 2$	$f_0 = 2e^0 - 2 = 0.0$
$x_1 = 0.1$	$y_1 = 2.010$	$f_1 = 2e^{0.1} - 2.010 = 0.20034$
$x_2 = 0.2$	$y_2 = 2.04$	$f_2 = 2e^{0.2} - 2.04 = 0.40281$
$x_3 = 0.3$	$y_3 = 2.09$	$f_3 = 2e^{0.3} - 2.09 = 0.60972$
$x_4 = 0.4$	$y_4 = ?$	

By Milne's Predictor formula

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$y_4^p = 2 + \frac{4(0.1)}{3}[2(0.20034) - (0.40281) + 2(0.60972)]$$

$$y_4^p = 2.16231$$

By Milne's Corrector formula

$$y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^p) \qquad f_4^p = 2e^{x_4} - y_4^p = 0.82134$$

$$y_4^c = 2.04 + \frac{0.1}{3}[0.40281 + 4(0.60972) + 0.82134]$$

$$y_4^c = 2.1620$$

To improve the accuracy of our results substitute the y_4^c in corrector formula

$$y_4^{c_1} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^c)$$

$$f_4^c = 2e^{x_4} - y_4^c = 0.82155$$

$$y_4^{c_1} = 2.04 + \frac{0.1}{3}[0.40281 + 4(0.60972) + 0.82155]$$

$$y_4^{c_1} = 2.16211$$

It is the required value of *y* at *x* = 0.4

3. Find y at x = 0.3, using applying Milne's method. Given $\frac{dy}{dx} = \frac{x+y}{2}$ and

x	-0.1	0	0.1	0.2
y	0.90878	1	1.11145	1.25253

Solution:- Construct the table by using given values

x	y	$y' = \frac{dy}{dx} = f(x,y) = \frac{x+y}{2}$
$x_0 = -0.1$	$y_0 = 0.90878$	$f_0 = 0.40439$
$x_1 = 0$	$y_1 = 1.0000$	$f_1 = 0.5$
$x_2 = 0.1$	$y_2 = 1.11145$	$f_2 = 0.605725$
$x_3 = 0.2$	$y_3 = 1.25253$	$f_3 = 0.72626$
$x_4 = 0.3$	$y_4 = ?$	?

By Milne's Predictor formula

$$y_4^p = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$$

$$y_4^p = 0.90878 + \frac{4(0.1)}{3}\left[2(0.5) - (0.60572) + 2(0.72626)\right]$$

$$y_4^p = 1.15502$$

By Milne's Corrector formula

$$y_4^c = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^p\right)$$

$$f_4^p = \frac{x_4 + y_4^p}{2} = 0.72751$$

$$y_4^c = 1.11145 + \frac{0.1}{3}\left[0.60572 + 4(0.72626) + 0.72751\right]$$

$$y_4^c = 1.25272$$

To improve the accuracy of our results substitute the y_4^c in corrector formula

$$y_4^{c_1} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^c\right)$$

$$f_4^c = \frac{x_4 + y_4^c}{2} = 0.77636$$

$$y_4^{c_1} = 1.11145 + \frac{0.1}{3}\left[0.60572 + 4(0.72626) + 0.77636\right]$$

$$y_4^{c_1} = 1.25435$$

It is the required value of *y* at *x* = 0.3

4. Find y at x = 4.4 using Milne's method. Given $5xy' - 2 + y^2 = 0$ and

x	4.0	4.1	4.2	4.3
y	1	1.0049	1.0097	1.0143

Solution:- Construct the table by using given values

x	y	$y' = \frac{dy}{dx} = f(x,y) = \frac{2-y^2}{5x}$
$x_0 = 4$	$y_0 = 1$	$f_0 = \frac{2-{y_0}^2}{5x_0} = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$f_1 = \frac{2-{y_1}^2}{5x_1} = 0.0485$
$x_2 = 4.2$	$y_2 = 1.0097$	$f_2 = \frac{2-{y_2}^2}{5x_2} = 0.0467$
$x_3 = 4.3$	$y_3 = 1.0143$	$f_3 = \frac{2-{y_3}^2}{5x_3} = 0.0452$
$x_4 = 4.4$	$y_4 = ?$	?

By Milne's Predictor formula

$$y_4^p = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$$

$$y_4^p = 1 + \frac{4(0.1)}{3}\left[2(0.0485) - (0.0467) + 2(0.0452)\right]$$

$$y_4^p = 1.01876$$

By Milne's Corrector formula

$$y_4^c = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^p\right)$$

$$f_4^p = \frac{2-\left(y_4^p\right)^2}{5x_4} = 0.04373$$

$$y_4^c = 1.0097 + \frac{0.1}{3}\left[0.0467 + 4(0.0456) + 0.04373\right]$$

$$y_4^c = 1.00909$$

To improve the accuracy of our results substitute the y_4^c in corrector formula

$$y_4^{c_1} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^c\right)$$

$$f_4^p = \frac{2-\left(y_4^c\right)^2}{5x_4} = 0.04462$$

$$y_4^{c_1} = 1.0097 + \frac{0.1}{3}\left[0.0467 + 4(0.0452) + 0.04462\right]$$

$$y_4^{c_1} = 1.01877$$

It is the required value of *y* at *x* = 4.4

5. Find y at x = 0.4 using Milne's method. Given

$$y' = xy + y^2,\ \ y(0)=1, y(0.1)=1.1169,$$
$$y(0.2)=1.2773, y(0.3)=1.5049$$

Solution: - Construct the table by using given values

x	y	$y' = f(x,y) = xy + y^2$
$x_0 = 0$	$y_0 = 1$	$f_0 = x_0 y_0 + y_0^2 = 1$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = x_1 y_1 + y_1^2 = 1.3592$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = x_2 y_2 + y_2^2 = 1.887$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = x_3 y_3 + y_3^2 = 2.7162$
$x_4 = 0.4$	$y_4 = ?$	?

By Milne's Predictor formula

$$y_4^p = y_0 + \frac{4h}{3}\left(2f_1 - f_2 + 2f_3\right)$$

$$y_4^p = 1 + \frac{4(0.1)}{3}\left[2(1.3592) - (1.887) + 2(2.7162)\right]$$

$$y_4^p = 1.8352$$

By Milne's Corrector formula

$$y_4^c = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^p\right)$$

$$f_4^p = x_4 y_4 + \left(y_4^p\right)^2 = 4.1020$$

$$y_4^c = 1.2773 + \frac{0.1}{3}\left[1.887 + 4(2.7162) + 4.102\right] = 1.8391$$

To improve the accuracy of our results substitute the y_4^c in corrector formula $y_4^{c_1} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^c\right)$ $\quad f_4^c = x_4 y_4 + \left(y_4^c\right)^2 = 4.1179$

$$y_4^{c_1} = 1.2773 + \frac{0.1}{3}\left[1.887 + 4(2.7162) + 4.1179\right]$$

$$y_4^{c_1} = 1.8396$$

It is the required value of *y* at *x* = 0.4

II . Adams Bashforth Predictor Corrector Method

Working rule:

Consider the initial value problem with a set of four points

$y(x_0) = y_0,\ y(x_1) = y_1, y(x_2) = y_2,\ y(x_3) = y_3,$ Here x_0, x_1, x_2, x_3 equally spaced. To find y4 at the point x4

Adams-Bashforth Predictor formula

$$y_4^p = y_3 + \frac{h}{24}\left(55f_3 - 59f_2 + 37f_1 - 9f_0\right)$$

Adams-Bashforth Corrector formula

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^p\right)$$

Where $f_4^p = \frac{dy}{dx} = f(x_4, y_4^p)$

To improve the accuracy again apply corrector formula by assuming

$$y_4^{c_1} = y_4^c$$

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^c\right)$$

Where $$f_4^c = \frac{dy}{dx} = f(x_4, y_4^c)$$

Example: 1

Find y(0.4), by applying Adams-Bashforth method given that $y' = \frac{xy}{2}$ and

x	0	0.1	0.2	0.3
y	1	1.0025	1.0101	1.0228

Solution:- Construct the table by using given values

x	y	$\frac{dy}{dx} = f(x,y) = \frac{xy}{2}$
$x_0 = 0$	$y_0 = 1$	$f_0 = 0$
$x_1 = 0.1$	$y_1 = 1.0025$	$f_1 = 0.0501$
$x_2 = 0.2$	$y_2 = 1.0101$	$f_2 = 0.1010$
$x_3 = 0.3$	$y_3 = 0.1762$	$f_3 = 0.1534$
$x_4 = 0.4$	$y_4 = ?$	

Adams-Bashforth Predictor formula

$$y_4^p = y_3 + \frac{h}{24}\left(55f_3 - 59f_2 + 37f_1 - 9f_0\right)$$

$$y_4^p = 1.0228 + \frac{0.1}{24}\left[55(0.1534) - 59(0.1010) + 37(0.05012)_1 - 9(0)\right]$$

$$y_4^p = 1.0408$$

Adams-Bashforth Corrector formula

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^p\right)$$

$$f_4^p = \frac{x_4 y_4^p}{2} = \frac{(0.4)(1.0408)}{2} = 0.2081$$

$$y_4^c = 1.0228 + \frac{0.1}{24}\left[0.0501 - 5(0.1010) + 19(0.1534) + 9(0.2081)\right]$$

$$y_4^c = 1.0408$$

Example. 2 Given $y' = x^2(1+y),\ y(1)=1, y(1.1)=1.233, y(1.2)=1.548,\ y(1.3)=1.979$ determine $y(1.4)$ by Adams- Bashforth method

Solution:- Construct the table by using given values

x	y	$y' = f(x,y) = x^2(1+y)$
$x_0 = 1$	$y_0 = 1$	$f_0 = x_0^2(1+y_0) = 2$
$x_1 = 1.1$	$y_1 = 1.233$	$f_1 = x_1^2(1+y_1) = 2.702$
$x_2 = 1.2$	$y_2 = 1.548$	$f_2 = x_2^2(1+y_2) = 3.669$
$x_3 = 1.3$	$y_3 = 1.979$	$f_3 = x_3^2(1+y_3) = 5.035$
$x_4 = 1.4$	$y_4 = ?$	?

Adams-Bashforth Predictor formula

$$y_4^p = y_3 + \frac{h}{24}\left(55f_3 - 59f_2 + 37f_1 - 9f_0\right)$$

$$y_4^p = 1.979 + \frac{0.1}{24}\left[55(5.035) - 59(3.669) + 37(2.702) - 9(2)\right]$$

$$y_4^p = 2.572$$

Adams-Bashforth Corrector formula

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^p\right)$$

$$f_4^p = x_4^2(1+y_4^p) = (1.4)^2(1+2.572) = 7.001$$

$$y_4^c = 1.979 + \frac{0.1}{24}\left[2.702 - 5(3.669) + 19(5.035) + 9(7.001)\right]$$

$$y_4^c = 2.575$$

To correct this solution again apply Adams-Bashforth Corrector formula,

Substitute y_4^c in $y_4^{c_1}$

$$y_4^{c_1} = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^c\right)$$

$$f_4^c = x_4^2(1+y_4^c) = (1.4)^2(1+2.575) = 7.007$$

$$y_4^{c_1} = 1.979 + \frac{0.1}{24}\left[2.702 - 5(3.669) + 19(5.035) + 9(7.007)\right]$$

$$y_4^{c_1} = 2.575$$

Example. 3 Given $\frac{dy}{dx} = 2e^x y$, $y(0)=2, y(0.1)=2.4725$, $y(0.2)=3.1261$, $y(0.3)=4.0524$ determine y(0.4) by Adams- Bashforth method.

Solution:- Construct the table by using given values

x	y	$y' = f(x,y) = 2e^x y)$
$x_0 = 0$	$y_0 = 1$	$f_0 = 4$
$x_1 = 0.1$	$y_1 = 2.4725$	$f_1 = 5.4652$
$x_2 = 0.2$	$y_2 = 3.1261$	$f_2 = 7.6364$
$x_3 = 0.3$	$y_3 = 4.0524$	$f_3 = 10.9406$
$x_4 = 0.4$	$y_4 = ?$	?

Adams-Bashforth Predictor formula

$$y_4^p = y_3 + \frac{h}{24}\left(55f_3 - 59f_2 + 37f_1 - 9f_0\right)$$

$$y_4^p = 4.0524 + \frac{0.1}{24}\left[55(10.9406) - 59(7.6364) + 37(5.4652) - 9(4)\right]$$

$$y_4^p = 5.3749$$

Adams-Bashforth Corrector formula

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^p\right)$$

$$f_4^p = 2y_4^p e^{x_4} = 2(5.3749)e^{0.4} = 16.0366$$

$$y_4^c = 4.0524 + \frac{0.1}{24}\left[5.4652 - 5(7.6364) + 19(10.9406) + 9(16.0366)\right]$$

$$y_4^c = 5.3835$$

To correct this solution again apply Adams-Bashforth Corrector formula,

Substitute y_4^c in $y_4^{c_1}$

$$y_4^{c_1} = y_3 + \frac{h}{24}\left(f_1 - 5f_2 + 19f_3 + 9f_4^c\right)$$

$$f_4^c = 2y_4^c e^{x_4} = 2(5.33835)e^{0.4} = 16.06248$$

$$y_4^{c_1} = 4.0524 + \frac{0.1}{24}\left[5.4652 - 5(7.6364) + 19(10.9406) + 9(16.0624)\right]$$

$$y_4^{c_1} = 5.3845$$

Example. 4

Solve the differential equation

$\frac{dy}{dx} = x - y^2$, at $x = 0.8$ given $y(0) = 0, y(0.2) = 0.02, y(0.4) = 0.0795, y(0.6) = 0.1762$

Using Adams- Bashforth method

Solution:- Construct the table by using given values

x	y	$\frac{dy}{dx} = f(x,y) = x - y^2$
$x_0 = 0$	$y_0 = 0$	$f_0 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$f_1 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$f_2 = 0.3936$
$x_3 = 0.6$	$y_3 = 0.1762$	$f_3 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	?

Adams-Bashforth Predictor formula

$$y_4^p = y_3 + \frac{h}{24}\left(55 f_3 - 59 f_2 + 37 f_1 - 9 f_0\right)$$

$$y_4^p = 0.1762 + \frac{0.2}{24}\left[55(0.5689) - 59(0.3936) + 37(0.1996) - 9(0)\right]$$

$$y_4^p = 0.30495$$

Adams-Bashforth Corrector formula

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5 f_2 + 19 f_3 + 9 f_4^p\right)$$

$$f_4^p = x_4 - \left(y_4^p\right)^2 = 0.8 - (0.3049)^2 = 0.70701$$

$$y_4^c = 0.1762 + \frac{0.2}{24}\left[0.1996 - 5(0.3936) + 19(0.56895) + 9(0.70701)\right]$$

$$y_4^c = 0.30457$$

To correct this solution again apply Adams-Bashforth Corrector formula,

Substitute y_4^c in $y_4^{c_1}$

$$y_4^{c_1} = y_3 + \frac{h}{24}\left(f_1 - 5 f_2 + 19 f_3 + 9 f_4^c\right)$$

$$f_4^c = x_4 - \left(y_4^p\right)^2 = 0.8 - (0.30457)^2 = 0.70724$$

$$y_4^c = 0.1762 + \frac{0.2}{24}\left[0.1996 - 5(0.3936) + 19(0.56895) + 9(0.70724)\right]$$

$$y_4^c = 0.30459$$

Example. 5

Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2}{1+y^2}, \quad \text{at } x = 1.0 \text{ given } y(0)=1, y(0.25)=1.0026,$$

$y(0.5)=1.0206, \; y(0.75)=1.0679$

Solution:- Construct the table by using given values

x	y	$\frac{dy}{dx} = f(x,y) = \frac{x^2}{1+y^2}$
$x_0 = 0$	$y_0 = 1$	$f_0 = 1$
$x_1 = 0.25$	$y_1 = 1.0026$	$f_1 = 0.0312$
$x_2 = 0.5$	$y_2 = 1.0206$	$f_2 = 0.1225$
$x_3 = 0.75$	$y_3 = 1.0679$	$f_3 = 0.2628$
$x_4 = 1.0$	$y_4 = ?$	?

Adams-Bashforth Predictor formula

$$y_4^p = y_3 + \frac{h}{24}\left(55 f_3 - 59 f_2 + 37 f_1 - 9 f_0\right)$$

$$y_4^p = 1.0679 + \frac{0.25}{24}\left[55(0.2628) - 59(0.1225) + 37(0.0312) - 9(0)\right]$$

$$y_4^p = 1.1552$$

Adams-Bashforth Corrector formula

$$y_4^c = y_3 + \frac{h}{24}\left(f_1 - 5 f_2 + 19 f_3 + 9 f_4^p\right)$$

$$f_4^p = \frac{x_4^2}{1+\left(y_4^p\right)^2} = \frac{1^2}{1+(1.1552)^2} = 0.4284$$

$$y_4^c = 1.0679 + \frac{0.25}{24}\left[0.0312 - 5(0.1224) + 19(0.2628) + 9(0.4284)\right]$$

$$y_4^c = 1.154$$

To correct this solution again apply Adams-Bashforth Corrector formula,

Substitute y_4^c in $y_4^{c_1}$

$$y_4^{c_1} = y_3 + \frac{h}{24}\left(f_1 - 5 f_2 + 19 f_3 + 9 f_4^c\right)$$

$$f_4^c = \frac{x_4^2}{1+\left(y_4^c\right)^2} = \frac{1^2}{1+(1.154)^2} = 0.4289$$

$$y_4^c = 1.0679 + \frac{0.25}{24}\left[0.0312 - 5(0.1224) + 19(0.2628) + 9(0.4289)\right]$$

$$y_4^c = 1.1541$$