MODULE 2: Gauss's law and Divergence, Energy and Potential, Conductors Dielectrics and Capacitance

2.0 Objectives

1. To Understand the concept of Potential and Potential Difference
2. To Learn the concepts of Energy density, current density
3. To derive current continuity equation
4. To understand the boundary Conditions

2.1 Energy expended in moving a point charge in an electric field

Electric field intensity is defined as the force experienced by unit test charge at a point p. If the test charge is moved against the electric field, then we have to exert a force equal and opposite to that exerted by the field and this requires work to be done.

Suppose we need to move a charge fo Q C a distance dl in an electric field E. The force on Q arising from the electric field is,

$$\mathbf{F}_E = Q\mathbf{E}$$

The differential amount of work done in moving charge Q over a distance dl is given by, $dW = -Q\mathbf{E}\cdot d\mathbf{L}$, as F =QE

Thus the work done to move the charge for the finite distance is given by,

$$W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E}\cdot d\mathbf{L}$$

2.3 Definition of Potential Difference and potential

Potential difference(V) is defined as the work done in moving unit positive charge from one point to another point in an electric field.

We know that,

$$W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E}\cdot d\mathbf{L}$$

Therefore V=W/Q= $\int_{\text{init}}^{\text{final}} \mathbf{E}\cdot d\mathbf{L}$

V_{AB} signifies potential difference between points A & B and the work done in moving the unit charge from B to A. Thus B is the initial point & A is the final point.

$$V_{AB} = -\int_{B}^{A} \mathbf{E}\cdot d\mathbf{L} \quad \text{V}$$

From the previous example, the work done in moving charge Q from ρ= b to ρ= a was,

$$W = \frac{Q\rho_L}{2\pi\epsilon_0}\ln\frac{b}{a}$$

Thus the potential difference between the points a & b is given by,

$$V_{ab} = \frac{W}{Q} = \frac{\rho_L}{2\pi\epsilon_0}\ln\frac{b}{a}$$

Absolute electric potential is defined as the work done in moving a unit positive charge from infinity to that point against the field.

Electric field is defined as force on unit charge.

E= F/Q.

By moving the charge Q aganist an electric field between the two points a & b work is done. Thus ,

Edl= Fxdl/Q =work/ charge.

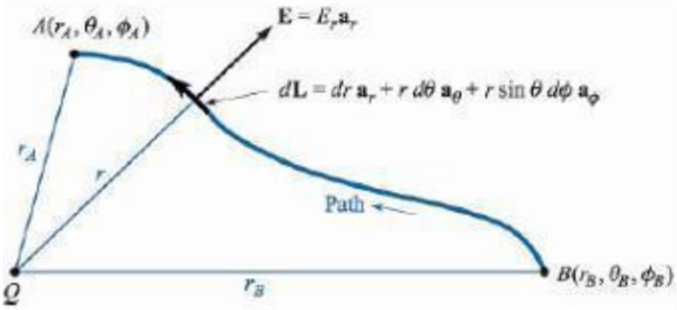
This work done per charge is the electric potential difference. Potential difference between points a and b at a radial distance of r_a and r_b from a point charge Q is given by, If the potential at point a is V_A and at point B is V_B, then

$$V_{AB} = V_A - V_B$$

Equipotential Surface is defined as "It is a surfacehaving the same value of potential" on composed of all- points such surfaces no work is charge, hence no potential difference involved in moving a unit between any two points on this surface.

2.4 Potential field of a point charge & system of charges

Consider a point charge Q to be placed in the origin of a spherical coordinate system. Consider 2 points A & B as shown in the figure.



Electric Potential difference between A & B, V_{AB} is given by,

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L}$$

dl in spherical co ordinate system is given the figure above and E=Q/ 4Π€r^2.
Therefore,

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

And

$$\boxed{V_{AB} = V_A - V_B}$$

Potential at a point has been defined as the work done in moving unit positive charge from zero reference to the point. Potential is independent of the path taken from one point to the other. Potential due to a single charge is given by

V(r)= Q1/ 4Π€R. If Q_1 is at r_1 & point p at r, then

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|}$$

Potential arising from 2 charges, Q_1 at r_1 and Q_2 at r_2, is given by

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_2|}$$

Potential due to n number of charges, is given by

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_2|} + \ldots + \frac{Q_n}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_n|}$$

Or

$$V(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_m|}$$

If point charge is a small element in the continuous volume charge distribution then,

$$V(\mathbf{r}) = \frac{\rho_v(\mathbf{r}_1)\Delta v_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|} + \frac{\rho_v(\mathbf{r}_2)\Delta v_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|} + \ldots + \frac{\rho_v(\mathbf{r}_n)\Delta v_n}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_n|}$$

As number of point charges in the volume charge distribution tends to infinity,

$$V(\mathbf{r}) = \int_{vol} \frac{\rho_v(\mathbf{r}')dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

Similarly if the point charges takes the form of a straight line then,

$$V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')dL'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

Similarly if the point charges takes the form of a surface charge then,

$$V(\mathbf{r}) = \int_S \frac{\rho_S(\mathbf{r}')dS'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

Potential is a function of inverse distance. Hence we can conclude that for a zero reference at infinity, then:

I Potential due to a single point charge is the work done in moving unit positive charge from zero reference to the point. Potential is independent of the path taken from one point to the other

II Potential field due to number of charges is the sum of the individual potential fields arising from each charge.

III. Potential due to continuous charge distribution is found by carrying a unit charge from infinity to the point under consideration.

$V_{AB} = V_A - V_B = -\int_B^A \mathbf{E}\cdot d\mathbf{L}$ is independent on the path chosen for the line integral, regardless of the source of the E field.

Hence we can conclude that no work is done in carrying a unit positive charge around any closed path, or

$$\oint \mathbf{E}\cdot d\mathbf{L} = 0$$

Any field that satisfies an equation of the form above is said to be conservative field

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2.5 Potential Gradient

Potential at any point is given by $V = -\int \mathbf{E}\cdot d\mathbf{L}$

Potential difference between 2 points separated by a very short length ΔL along which E is essentially constant, is given by

$$\Delta V \doteq -\mathbf{E}\cdot\Delta\mathbf{L}$$

In rectangular co ordinate system,

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

, As V is a unique function of x,y,z. Then,

$$dV = -\mathbf{E}\cdot d\mathbf{L} = -E_x\,dx - E_y\,dy - E_z\,dz$$

Since both the expressions are true with respect dx,dy & dz, we can write

$$E_x = -\frac{\partial V}{\partial x}$$
$$E_y = -\frac{\partial V}{\partial y}$$
$$E_z = -\frac{\partial V}{\partial z}$$

Therefore,

$$\boxed{\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right)}$$

In rectangular co ordinate system,

$$\boxed{\text{grad } V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z}$$

Combining all the above equations allows us to use a compact expression that relates E & V, $\boxed{\mathbf{E} = -\nabla V}$

Gradient in other coordinate system is as given below,

$$\boxed{\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cylindrical)}}$$

$$\boxed{\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \qquad \text{(spherical)}}$$

Given the potential field, $V = 2x^2y - 5z$, and a point $P(-4, 3, 6)$, we wish to find several numerical values at point P: the potential V, the electric field intensity $\mathbf{E}$, the direction of $\mathbf{E}$, the electric flux density $\mathbf{D}$, and the volume charge density ρ_v.

Solution. The potential at $P(-4, 5, 6)$ is

$$V_P = 2(-4)^2(3) - 5(6) = 66 \quad \text{V}$$

Next, we may use the gradient operation to obtain the electric field intensity,

$$\mathbf{E} = -\nabla V = -4xy\mathbf{a}_x - 2x^2\mathbf{a}_y + 5\mathbf{a}_z \quad \text{V/m}$$

The value of $\mathbf{E}$ at point P is

$$\mathbf{E}_P = 48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z \quad \text{V/m}$$

and

$$|\mathbf{E}_P| = \sqrt{48^2 + (-32)^2 + 5^2} = 57.9 \quad \text{V/m}$$

The direction of $\mathbf{E}$ at P is given by the unit vector

$$\mathbf{a}_{E,P} = (48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z)/57.9$$
$$= 0.829\mathbf{a}_x - 0.553\mathbf{a}_y + 0.086\mathbf{a}_z$$

If we assume these fields exist in free space, then

$$\mathbf{D} = \epsilon_0\mathbf{E} = -35.4xy\mathbf{a}_x - 17.71x^2\mathbf{a}_y + 44.3\mathbf{a}_z \quad \text{pC/m}^3$$

6 Energy Density in an Electric Field

Consider a surface without charge. Bringing a charge Q1 from infinity to any point on the surface requires no work as there is no field present. The positioning of Q2 at appoint in the field of Q1 requires an amount of work to be done which is given by

$$\text{Work to position } Q_2 = Q_2 V_{2,1}$$

.

Similarly work required to position each additional charge in the field is given by,

$$\text{Work to position } Q_3 = Q_3 V_{3,1} + Q_3 V_{3,2}$$
$$\text{Work to position } Q_4 = Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3}$$

Total positioning work = Potential energy of the field

$$= W_E = Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2} + Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3} + \dots$$

Bringing the charges in the reverse order, the work done is given by,

$$W_E = Q_1 V_{1,2} + Q_1 V_{1,3} + Q_2 V_{2,3} + Q_1 V_{1,4} + Q_2 V_{2,4} + Q_3 V_{3,4} + \dots$$

Adding the 2 energy expressions, we get

$$2W_E = Q_1(V_{1,2} + V_{1,3} + V_{1,4} + \dots) + Q_2(V_{2,1} + V_{2,3} + V_{2,4} + \dots) + Q_3(V_{3,1} + V_{3,2} + V_{3,4} + \dots)$$

For n number of charges,

$$W_E = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + \dots) = \tfrac{1}{2} \sum_{m=1}^{m=N} Q_m V_m$$

2.7 Potential energy in a continuous charge distribution:

For the region with continuous charge distribution, the equation for WE=
By vector identity which is true for any scalar function V & vector D,

$$\nabla \cdot (V\mathbf{D}) \equiv V(\nabla \cdot \mathbf{D}) + \mathbf{D} \cdot (\nabla V)$$

,

Then,

$$W_E = \tfrac{1}{2} \int_{\text{vol}} \rho_v V dv = \tfrac{1}{2} \int_{\text{vol}} (\nabla \cdot \mathbf{D}) V \, dv$$
$$= \tfrac{1}{2} \int_{\text{vol}} [\nabla \cdot (V\mathbf{D}) - \mathbf{D} \cdot (\nabla V)] \, dv$$

.

From Guass law, We can write

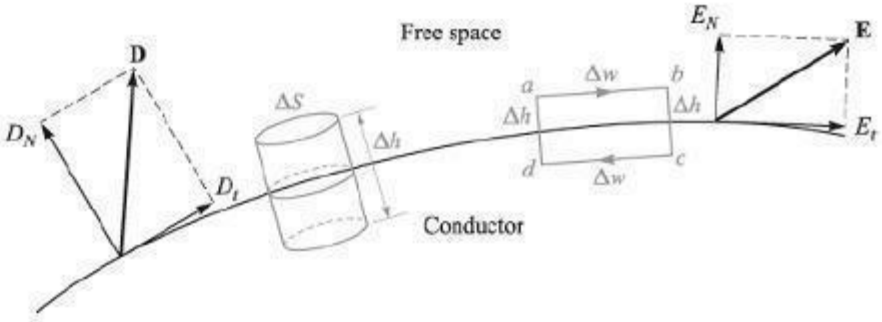
$$W_E = \frac{1}{2}\oint_S (V\mathbf{D}) \cdot d\mathbf{S} - \frac{1}{2}\int_{\text{vol}} \mathbf{D} \cdot (\nabla V)\, dv$$

and from gradient

$$W_E = \frac{1}{2}\int_{\text{vol}} \mathbf{D} \cdot \mathbf{E}\, dv = \frac{1}{2}\int_{\text{vol}} \epsilon_0 E^2\, dv$$

2.8 Boundary condition for conductor free space interface:

Consider a closed path at the boundary between conductor and a dielectric, such that Δh→0.



We know that work done in moving a charge over a closed path is zero i.e.,

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

Therefore the integral can be broken up as,

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

Let the length from a to b or c to d be ΔW and from a to d or b to c be Δh , hence we obtain,

$$E_t \Delta w - E_{N,\text{at } b} \frac{1}{2}\Delta h + E_{N,\text{at } a} \frac{1}{2}\Delta h = 0$$

. Hence we obtain EΔW=0 & therefore E_t=0

Hence at the conductor dielectric interface tangential component of the electric field intensity is zero.

Consider a gaussian cylinder of radius ρ and height Δh at the boundary, Applying Gauss law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

& then integrating over the distinct surfaces we get

$$\int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{sides}} = Q$$

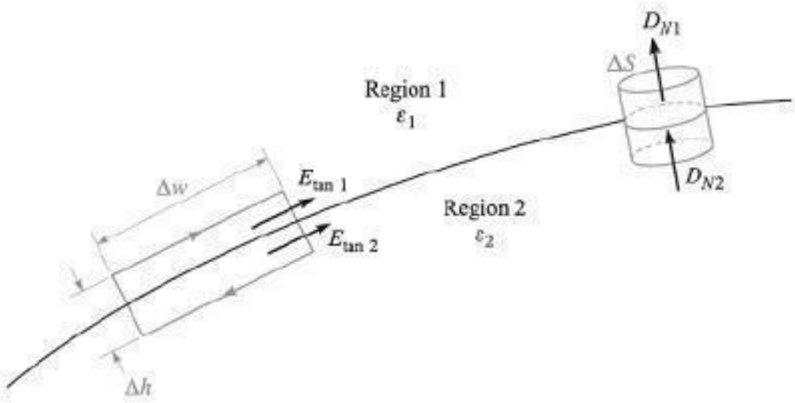
Flux experienced by the lateral surface is zero & Flux experienced by the bottom surface is zero as charge inside the conductor is zero. Therefore

$$D_N \Delta S = Q = \rho_S \Delta S \quad \text{or} \quad D_N = \rho_S$$

At the conductor dielectric interface normal component of the electric flux density is equal to the surface charge density.

2.8 Boundary condition for perfect dielectric:

Consider a closed path abcda at the dielectric dielectric interface & Δh→0. The work done in moving a unit charge over a closed path is zero. Therefore,



We know that the work done in moving a unit charge over a closed path is zero. Therefore,

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

, and hence

$$E_{\tan 1}\,\Delta w - E_{\tan 2}\,\Delta w = 0$$

.

The small contribution of the normal component of E due to Δh becomes negligible. Therefore,

$E_{\tan 1} = E_{\tan 2}$. & as D = € E we get,

$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2} \quad \text{or} \quad \frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2}$$

.

At the dielectric – dielectric boundary tangential component of the E is continuous where as tangential component of electric flux density is discontinuous.

Consider a gaussian cylinder of radius ρ and height Δh at the boundary, Applying Gauss law, & then integrating over the distinct surfaces we get

$$\int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{sides}} = Q$$

. Flux experienced by the lateral surface is zero. Therefore

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_S \Delta S$$

.

From which,

$$\boxed{D_{N1} - D_{N2} = \rho_S}$$

.

For perfect dielectric, DN1= DN2, then €2E2 = €1E1.

At the dielectric dielectric boundary normal component of the flux density is continuous. Normal components of D are continuous,

$$D_{N1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N2}$$

. The ratio of the tangential components,

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

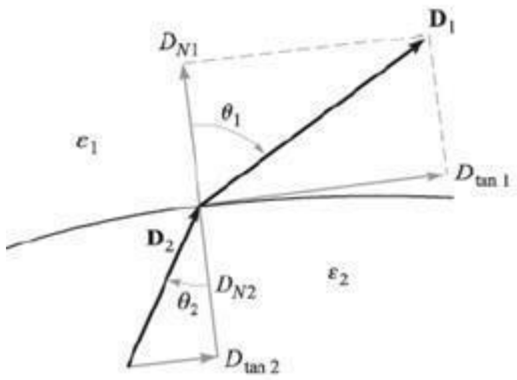
Or $\epsilon_2 D_1 \sin\theta_1 = \epsilon_1 D_2 \sin\theta_2$.

And

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2}.$$

The magnitude of D is given by,

$$D_2 = D_1 \sqrt{\cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1}.$$



Out comes

At the end of the unit the students are able to understand the concepts of Potential and Potential difference, energy and current densities, current continuity equation, and different boundary conditions.

Recommended questions

1. Define electric scalar potential. Establish the relationship between intensity and potential.
2. Discuss the boundary conditions between 2 perfect dielectrics.
3. State & explain the principle of charge conservation.
4. Derive for energy stored in an electrostatic field.

5. Derive for energy expended in moving a point charge in an electric field.

6. Define Potential & potential difference.

7. Prove that E is Grad of V

8. Write a short note on dipole

9. Three point charges, 0.4 μC each, are located at (0,0,-1), (0,0,0) and (0,0,1) in free space.
(a). Find an expression for the absolute potential as a function of Z along the lne x=0, y=1.
(b) Sketch V(Z).

Further Reading

1. Energy Electromagnetics, William H Hayt Jr. and John A Buck, Tata McGraw-Hill, 7th edition,2006.
2. Electromagnetics with Applications, John Krauss and Daniel A Fleisch McGraw-Hill, 5th edition, 1999
3. Electromagnetic Waves And Radiating Systems, Edward C. Jordan and Keith G Balmain, Prentice – Hall of India / Pearson Education, 2nd edition, 1968.Reprint 2002
4. Field and Wave Electromagnetics, David K Cheng, Pearson Education Asia, 2nd edition, 1989, Indian Reprint – 2001