1.1 Experimental Law of Coulomb

Coulomb’s law states that the electrostatic force $F$ between two point charges $q_1$ and $q_2$ is directly proportional to the product of the magnitude of the charges, and inversely proportional to the square of the distance between them, and it acts along the line joining the two charges. Then, as per the Coulomb’s Law,

\[
F \propto k |q_1 q_2|
\]

Or

\[
F = \frac{k |q_1 q_2|}{r^2} \text{ N}
\]

Where $k$ is the constant of proportionality whose value varies with the system of units. $\hat{r}$ is the unit vector along the line joining the two charges.

In SI unit, $k = \frac{1}{4\pi \varepsilon_0}$.

Where $\varepsilon_0$ is called the permittivity of the free space.
It has an assigned value given as $\varepsilon_0 = 8.83 \times 10^{-12}$ F/m.
**Force on a point charge:**
The forces of attraction/repulsion between two point charges \( \mathcal{Q}_1 \) and \( \mathcal{Q}_2 \) (charges whose size is much smaller than the distance between them) are given by Coulomb’s law:

\[
\mathbf{F}_1 = k \cdot \frac{\mathcal{Q}_1 \mathcal{Q}_2}{R^2} \mathbf{a}_{21}
\]

\[
\mathbf{F}_2 = k \cdot \frac{\mathcal{Q}_1 \mathcal{Q}_2}{R^2} \mathbf{a}_{12}
\]

where \( k \approx 9 \times 10^9 \) m/N in SI units, and \( R \) is the distance between the two charges. Here, \( \mathbf{F}_1 \) is the force exerted on \( \mathcal{Q}_1 \), and \( \mathbf{F}_2 \) is the force acting on \( \mathcal{Q}_2 \). The unit vector \( \mathbf{a}_{21} \) points from charge 2 toward charge 1. Accordingly, \( \mathbf{a}_{12} = -\mathbf{a}_{21} \).

Force on \( Q_1 \) is given by

\[
\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\mathcal{Q}_1 \mathcal{Q}_2}{R^2} \mathbf{a}_{21}
\]

Force due to several charges

Let there be many point charges \( q_1, q_2, q_3, \ldots, q_n \) at distances \( r_1, r_2, r_3, \ldots, r_n \) from charge \( q \). The force \( F_1, F_2, F_3, \ldots, F_n \) at the charges \( q_1, q_2, q_3, \ldots, q_n \) respectively are:

\[
q \left( \frac{q_1}{4\pi \varepsilon_0 r_1^2} \mathbf{\hat{r}}_1 + \frac{q_2}{4\pi \varepsilon_0 r_2^2} \mathbf{\hat{r}}_2 + \cdots \right)
\]

\[
F = F_1 + F_2 + F_3 + \ldots
\]

Hence,

\[
F = q \left( \frac{1}{4\pi \varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i^2} \mathbf{\hat{r}}_i \right) N
\]
1.1 Objectives

After going through this section, the students are able to:
1. State Coulomb's law
2. Application of Coulomb's Law to point charge as well as several charges

1.2 Electric field intensity

Electric field intensity at any point in an electric field is the force experienced by positive unit charge placed at that point.

Consider a charge \( Q \) located at a point A. At the point B in the electric fields set up by \( Q \), it is required to find the electric field intensity \( E \).

Let the charge at B be \( \Delta q \) and let the charge \( Q \) be fixed at A. Let \( r \) be the distance between A and B. As per the Coulomb's Law, the force between \( Q \) and \( q \) is given by:

\[
F = \frac{Q \Delta q}{(4 \pi \varepsilon_0 r^2)} \hat{r} \text{ N}
\]

If it is a unit positive charge, then by definition, \( F \) in the above equation gives the magnitude of the electric field intensity \( E \).

i.e. \( E = F \) when \( \Delta q = 1 \).

Therefore, the magnitude of the electric field strength is:

\[
E = \frac{Q}{(4 \pi \varepsilon_0 r^2)} \text{ V/m}
\]

Let \( r \) be the unit vector along the line joining A and B. Thus, the vector relation between \( E \) is written as:

\[
E = \frac{Q}{(4 \pi \varepsilon_0 r^2)} \hat{r} \text{ V/m}
\]

1.2.1 Electric Field intensity due to several charges

Let there be many point charges \( q_1, q_2, q_3, \ldots \) at distances \( r_1, r_2, r_3 \ldots \) \( \ldots \) \( r_n \) be the corresponding unit vectors. The field \( E_1, E_2, E_3 \ldots \) at the charges \( q_1, q_2, q_3 \ldots \ldots \) \( q_n \) respectively are:

\[
E = Eq_1 + Eq_2 + Eq_3 + \ldots
\]

\[
E = \frac{1}{4\pi \varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i^2} \hat{r}_i
\]

Hence,
1.2.2 Electric field intensity at a point due to an infinite sheet of charge

Let us assume a straight line charge extending along Z axis in a cylindrical coordinate system from $-\infty$ to $+\infty$ as shown in the figure 1.1. Consider an incremental length $dl$ at a point on the conductor. The incremental length has an incremental charge of $dQ = \rho l dl = \rho ldz'$ Coulombs. Considering the charge $dQ$, the incremental field intensity at point $p$ is given by,

\[ dE = \frac{\rho_l dz'(r - r')}{4\pi\epsilon_0 |r - r'|^3} \]

Where

\[ r = ya_y = \rho a_\rho \]
\[ r' = z'a_z \]

and

\[ r - r' = \rho a_\rho - z'a_z \]

Therefore,

\[ E_\rho = \int_{-\infty}^{\infty} \frac{\rho_l \rho dz'}{4\pi\epsilon_0 (\rho^2 + z'^2)^{3/2}} \]

Integrating the above and substituting $z' = \rho \cot \theta$, we get
and

\[ E_φ = \frac{ρL}{2πε₀ρ} \]

### 1.2.3 Electric field intensity at a point due to a infinite sheet of charge:

Let us assume a infinite sheet of charge with surface charge density \( ρ_s \) as shown in the figure 1.2. Divide the sheet of charge into differential width strips. number of strip Consider an incremental length \( dl \) at a point on the conductor. The line charge density \( ρ_l = ρ_s \, dy' \).

![Diagram of an infinite sheet of charge](image)

The differential Electric field intensity at point P,

\[ dE_x = \frac{ρ_s \, dy'}{2πε₀\sqrt{x^2 + y'^2}} \cos θ = \frac{ρ_s}{2πε₀} \frac{x \, dy'}{x^2 + y'^2} \]

adding the effects of all the strips,

\[ E_x = \frac{ρ_s}{2πε₀} \left[ \int_{-∞}^{∞} \frac{x \, dy'}{x^2 + y'^2} \right] = \frac{ρ_s}{2πε₀} \left[ \tan^{-1} \frac{y'}{x} \right]_{-∞}^{∞} = \frac{ρ_s}{2ε₀} \]

Therefore,

\[ E = \frac{ρ_s}{2ε₀} a_N \]
### 1.2.4 Electric field at a point on the axis of charged circular ring:

Let \( \rho \) be the charge density of the ring.

So,

\[
\rho = \frac{dq}{dl} \\
\Rightarrow dq = \rho dl
\]

Electric field due to an infinitely small element = \( \text{d}E = \frac{dq}{4\pi \varepsilon_0 r^2} \)

where \( r^* \) is the unit vector along AP.

dE can be resolved into two rectangular components, \( \text{d}E_x \) and \( \text{d}E_y \). Now, \( \text{d}E_x = \text{d}E \cos \theta \).

Taking the magnitude of \( \text{d}E \) from above, the equation becomes,

\[
\frac{dq \cos \theta}{4\pi \varepsilon_0 r^2}
\]

\( \text{d}E_x = \frac{x}{r} \cos \theta = \frac{x}{r} \)

Substituting for \( dq \) from above, we have;

\[
\text{d}E_x = \frac{\rho dl}{4\pi \varepsilon_0 r^2}
\]

The component \( \text{d}E_y \) is directed downwards. If we consider an element of the ring at a point diametrically opposite to A, then its \( \text{d}E_y \) component points upwards and hence, cancels with that due to element A. The \( \text{d}E_x \) components add up.

\[
\int \text{d}E_y = 0.
\]

The total field at P is the sum of the fields due to all the elements of the ring.

Therefore,

\[
E = \int \text{d}E = \int \text{d}E_x + \int \text{d}E_y = \int \text{d}E_x
\]

\[
E = \int \text{d}E_x = \frac{\rho x}{4\pi \varepsilon_0 r^2} \int_0^{2\pi} dl = \frac{\rho x (2\pi R)}{4\pi \varepsilon_0 r^2}
\]

But,

\[
r = (R^2 + x^2)^{1/2}
\]

Therefore,

\[
E = \frac{2\pi \rho \alpha x}{4\pi \varepsilon (R^2 + x^2)^{3/2}}
\]

Where, \( \alpha x \) is the unit vector along the x axis.
1.2 Objectives

At the end of this section the students are able to
1. Define Electric field Intensity
2. Derive Electric field intensity at a due to several charges
3. Derive Electric field Intensity at a point due to sheet of charge
4. Derive Electric field intensity at a point on the axis of charged circular ring

1.3 Electric flux:

The concept of electric flux is useful in association with Gauss' law. The electric flux through a planar area is defined as the electric field times the component of the area perpendicular to the field. If the area is not planar, then the evaluation of the flux generally requires an area integral since the angle will be continually changing.

When the area A is used in a vector operation like this, it is understood that the magnitude of the vector is equal to the area and the direction of the vector is perpendicular to the area.

Consider a concentric sphere having radius of ‘a’ m charged up to +Q C. This sphere is then placed in another sphere having a radius of ‘b’ m as shown in the figure 1.4.

![Electric Flux Diagram](image)

There is no electrical connection between them. The outer sphere is momentarily charged, then it found that the charge on the outer sphere is equal to the charge on the inner sphere. This is depicted by the radial lines. This is referred as displacement flux. Therefore, \( \Psi = Q \).

1.3.1 Electric flux density:

If +Q C of charge on the inner sphere produces the electric flux of \( \psi \), then electric flux \( \psi \) uniformly distributed over the surface area \( 4\pi a^2 \text{ m}^2 \), where a is the radius of the inner sphere.
The electric flux density is given by

\[ D|_{r=a} = \frac{Q}{4\pi a^2} a_r \quad \text{(inner sphere)} \]

Similarly for the outer sphere,

\[ D|_{r=b} = \frac{Q}{4\pi b^2} a_r \quad \text{(outer sphere)} \]

If the inner sphere becomes smaller and smaller retaining a charge of Q C, it becomes a point charge. The flux density at point ‘r’ from the point charge is given by,

\[ D = \frac{Q}{4\pi r^2} a_r \]

The electric field intensity due to point charge in free space is given by,

\[ E = \frac{Q}{4\pi \epsilon_0 r^2} a_r \]

Therefore in free space,

\[ D = \epsilon_0 E \]

### 1.3 Objective

After going through this section the students should be able to
1. Define Electric flux
2. Explain Electric flux density

### 1.4 Gauss law:

The Gauss's law states that. "The electric flux passing through any closed surface is equal to the total charge enclosed by the surface"

For the Gaussian-surface shown in the following figure, the Gauss' law can be expressed mathematically, .

\[ \Psi = \oint \vec{D}_s \cdot d\vec{s} = Q \]

Where
\[ \Psi = \text{flux passing through the closed surface} \]

\[ \int_s = \text{surface integral} \]

\[ D_s = \text{flux density (vector quantity) normal to the surface} \]

\[ Q = \text{Total charge enclosed in the surface} \]

Gauss law for charge \( Q \) enclosed in a closed surface:

Let \( Q \) be the point charge placed at the origin of imaginary sphere in spherical coordinate system with a radius of "a" as illustrated in the figure. The electrical field intensity of the point charge is found to be equal to

\[ \vec{E} = \frac{Q}{4\pi \varepsilon_0 r^2} \hat{a}_r \quad \text{(1)} \]

Where \( r = \text{C}1 \)

and we also know that the relation between \( E \) and \( D \) as,

\[ \vec{D} = \varepsilon_0 \vec{E} \quad \text{(2)} \]
Therefore from (1) and (2) we get.

\[
\vec{D} = \varepsilon_0 \frac{Q}{4\pi \frac{\varepsilon_0}{\varepsilon_0} r^2} \vec{a}_r
\]

\[
= \frac{Q}{4\pi a^2} \vec{a}_r
\]

at the surface of the sphere,

\[
\vec{D} = \frac{Q}{4\pi a^2} \vec{a}_r
\]

The differential element of area on a spherical surface is, in spherical coordinate form is given by,

\[
ds = r^2 \sin \theta \ d\theta \ d\phi = a^2 \sin \theta \ d\theta \ d\phi
\]

Or

\[
d\vec{s} = a^2 \sin \theta \ d\theta \ d\phi \vec{\vec{a}}_r
\]

Then the required integrand

\[
= \vec{D}_s \cdot d\vec{s}
\]

\[
= \frac{Q}{4\pi a^2} \cdot a^2 \sin \theta \ d\theta \ d\phi \vec{\vec{a}}_r \cdot \vec{\vec{a}}_r
\]

\[
= \frac{Q}{4\pi} \sin \theta \ d\theta \ d\phi \quad (\because \vec{\vec{a}}_r \cdot \vec{\vec{a}}_r = 1 \text{ from vector basics})
\]

Then the integration over the surface as required for Gauss' law.

\[
\oint_{\Sigma} \vec{D}_s \cdot d\vec{s} = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{Q}{4\pi} \sin \theta d\theta d\phi
\]

The limits placed for integral indicate that the integration over the entire sphere in spherical co-ordinate system on integration we get
Thus we get, comparing LHS of Gauss’ law as

\[ \psi = Q \]

This indicates that, Q coulombs of electric flux are crossing the surface as the enclosed charge is Q coulombs.

1.4.1 Application of Gauss law:

In case of asymmetry, we need to choose a very closed surface such that D is almost constant over the surface. Consider any point P shown in the figure 1.6 located in the rectangular co-ordinate system.

The value of D at point P, may be expressed in rectangular components as,

\[ D = D_x \hat{a}_x + D_y \hat{a}_y + D_z \hat{a}_z \]. From Gauss law, we have

\[ \int_S D \cdot dS = Q \]

In order to evaluate the integral over the closed surface, the integral must be broken into six integrals, one over each surface,

\[ \int_S D \cdot dS = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{top}} + \int_{\text{bottom}} \].

The surface element is very small & hence D is essentially constant,
\[
\int_{\text{front}} + \int_{\text{back}} = \frac{\partial D_x}{\partial x} \Delta x \Delta y \Delta z,
\]
Similarly,
\[
\int_{\text{right}} + \int_{\text{left}} = \frac{\partial D_y}{\partial y} \Delta x \Delta y \Delta z
\]
and,
\[
\int_{\text{top}} + \int_{\text{bottom}} = \frac{\partial D_z}{\partial z} \Delta x \Delta y \Delta z
\]
Therefore collectively,
\[
\oint_S \mathbf{D} \cdot d\mathbf{S} = \left( \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \Delta y \Delta z
\]
or
\[
\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \left( \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v
\]
Charge enclosed in volume \(\Delta v\),

\[
\text{Charge enclosed in volume } \Delta v \doteq \left( \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times \text{volume } \Delta v
\]

### 1.4 Objectives

At the end of this section the students are able to
1. State and prove Gauss Law
2. Apply Gauss law to find the charge enclosed in differential volume

### 1.5 Divergence:

From Gauss law, we know that,
\[
\left( \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \doteq \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \frac{Q}{\Delta v}
\]
And applying limits,
The last term in the equation is the volume charge density, $\rho_v$.

$$\left( \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v$$

We shall write it as two separate equations,

$$\left( \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \rho_v$$

Divergence is defined as,

$$\text{div} \mathbf{D} = \left( \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right)$$

Divergence of $\mathbf{A} = \text{div} \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}.$

Statement: The flux crossing the closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume, as the volume shrinks to zero.

Divergence in Cartesian system,

$$\text{div} \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \quad \text{(cartesian)}$$

Divergence in Cylindrical system,

$$\text{div} \mathbf{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \phi) + \frac{1}{\rho} \frac{\partial \phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \quad \text{(cylindrical)}$$

Divergence in Spherical system,

$$\text{div} \mathbf{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta D_\theta) + \frac{1}{r \sin \theta} \frac{\partial D_\phi}{\partial \phi} \quad \text{(spherical)}$$
1.6.1 Maxwell’s First equation:

From divergence theorem, we have

\[
\text{div } D = \lim_{\Delta v \to 0} \frac{\oint_S A \cdot dS}{\Delta v}
\]
\[
\text{div } D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}
\]
\[
\text{div } D = \rho_v
\]

From Gauss law,

\[
\oint_S A \cdot dS = Q
\]

Per unit volume,

\[
\frac{\oint_S A \cdot dS}{\Delta v} = \frac{Q}{\Delta v}
\]

As the volume shrinks to zero,

\[
\lim_{\Delta v \to 0} \frac{\oint_S A \cdot dS}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}
\]

Therefore,

\[
\text{div } D = \rho_v
\]

1.6.2 Divergence theorem:

The del operator is defined as a vector operator.

\[
\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z
\]

In Cartesian coordinate system,

\[
\nabla \cdot D = \frac{\partial}{\partial x} (D_x) + \frac{\partial}{\partial y} (D_y) + \frac{\partial}{\partial z} (D_z)
\]

Which is equal to,

\[
\nabla \cdot D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}
\]
Therefore,

\[
\text{div } \mathbf{D} = \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}.
\]

From Gauss law, we have

\[
\int_S \mathbf{D} \cdot d\mathbf{S} = Q
\]

And by letting,

\[
Q = \int_{\text{vol}} \rho_v dv \quad \text{&} \quad \nabla \cdot \mathbf{D} = \rho_v.
\]

Hence we have,

\[
\int_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \nabla \cdot \mathbf{D} dv
\]

1.6 Objectives

At the end of this section the students are able to

1. Explain the concept of divergence
2. Derive Maxwell’s First Equation
3. State and prove Divergence theorem
4. 
5.

1.7 Recommended Questions

1. State Coulomb’s law of force between any 2 point charges & indicate the units of the quantities involved.

2. Derive the general expression for electric field vector due to infinite line charge using Gauss law.


4. Derive the general expression for E at a height h(h<a) , along the axis of the ring charge & normal to its plane.

5. From gauss law show that \( \mathbf{D} = \sigma \mathbf{v} \)
6. State and prove divergence theorem for symmetric condition.

7. State and prove divergence theorem for asymmetric condition

1.8 Further Readings


